GENERALIZED ABSOLUTE SUMMABILITY WITH INDICES AND THEIR APPLICATIONS

DISSERTATION

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF

Master of Philosophy

IN

APPLIED MATHEMATICS

BY

TARIQ AHMED GHESTI

DEPARTMENT OF APPLIED MATHEMATICS
FACULTY OF ENGINEERING & TECHNOLOGY
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)

1993

To My Parents

in esteem and affection

C E R T I F I C A T E

Certified that **Mr. Tariq Ahmed Chesti** has carried out the research on **"GENERALIZED ABSOLUTE SUMMABILITY WITH INDICES AND THEIR APPLICATIONS"** under my supervision and work is suitable for submission for the award of the degree of Master of Philosophy in "Applied Mathematics".

AAKhan

(DR. Abrar Ahmad Khan)

Supervisor

ACKNOWLEDGEMENTS

All the thanks are due to almighty **Allah**, who bestowed upon me the capability to achieve this target.

I express my heartful and sincere gratitude to my supevisor **Dr. Abrar Ahmad Khan,** Reader, Department of Applied Mathematics, for uninstincted help, encouragement, keen interest and valuable suggestions throughout this work.

It is my privilege to take this opportunity to acknowledge my warmest thanks to **Prof. Z.U. Ahmad,** M.Sc., D.Phil., D.Sc. and **Dr. Mursaleen** for their valuable guidance and enthusiastic direction, due to which I could complete the work of my M.Phil. dissertation.

My sincere thanks are due to **Prof. Tariq Aziz,** Chairman, Depatment of Applied Mathematics, for providing necessary facilities.

I also take this opportunity to place on record my deepest sense of gratitude to my parents for their patience, kind help and constant encouragement during the preparation of this work.

A special token of deep appreciation to all my friends and colleagues especially to **Mr. Pirzada Sharief - ud-din** and **Mr. Neyaz Ahmad Sheikh** for their generous help and useful suggestions.

Finally, I must offer my thanks to **Mr. Zaki** who has taken all the troubles and pains to type it so nicely, in a short time.

Tariq Ahmad

TARIQ AHMAD CHESTI

CONTENTS

-o-o-o-o-o-

CHAPTER 0

CHAPTER 0

NOTE ON CONVENTIONS

Here we state a few conventions which are not emphasized in the following Chapters.

0.1. SUMMATION CONVENTIONS:

By $\sum_{\alpha}^{\beta} f(n)$ we mean the sum of all values of $f(n)$ for which $\alpha \leq n \leq \beta$; if $\beta < \alpha$, is zero. Summations are over $0, 1, 2, \ldots$, where there is no indications to the contrary. Σ usually denotes $\sum_{0}^{\infty}$ unless the first term is indeterminate, in which case it will denote $\sum_{1}^{\infty}$.

0.2. BINOMIAL COEFFICIENTS:

For $n = 0, 1, 2, \ldots, A_n^{\alpha}$ is defined by the identity:

$$\Sigma A_n^{\alpha} x^n = (1-x)^{-\alpha-1} \qquad (|x| < 1),$$

$$A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!}, \quad (\alpha > -1);$$

$$\sim \frac{n^{\alpha}}{\Gamma(n+1)} \qquad (\alpha \neq -1, -2, \ldots)$$

$$A_n^{\alpha} = \begin{cases} 1 & (n=0,\ \alpha \text{ real}) \\ 0 & (n<0,\ \alpha \text{ real}) \end{cases}; \quad A_n^{-\alpha} = 0 \quad (n > \alpha, \alpha = 1, 2, \ldots)$$

and for $\alpha \leq -1$,

$$\Sigma \, |A_n^{\alpha}| < \infty.$$

0.3. THE LETTER K:

K denotes, throught, an absolute constant, independent of the variable under consideration, but is not necessarily the same at each occurrence.

K is also used as a symbol for the word 'Conservative'.

0.4. SYMBOLS T, AK, AT:

T, AK and AT are used to symbolize the word 'regular' and the phrases, 'absolutely conservative' and 'absolutely regular' respectively.

0.5. ORDER NOTATION O AND o:

If g is a positive function of a variable which tends to a given limit, we shall write

$$f = o(g),$$

if $f/g \longrightarrow 0$, and

$$f = O(g),$$

if $|f|/g < K$. In particular, $f = o(1)$, means that $f \longrightarrow 0$, and $f = O(1)$ means that f is bounded.

0.6. SYMBOLS $\subseteq$, $\subset$ AND $\sim$:

Given two methods of summability (or absolute summability or absolute summability with index), P and Q, we write $P \subseteq Q$, or $Q \supseteq P$ for 'P is included in Q' or 'Q includes P' to mean that every sequence summable by P is also summable by Q.

If $P \subseteq Q$ and $Q \subseteq P$, the two methods P and Q are said to be equivalent and we write $P \sim Q$.

If $P \subseteq Q$ and there exists a sequence which is summable Q but not summable P, then we write $P \subset Q$.

0.7. FINITE DIFFERENCES:

For any sequence $\{f_n\}$, we write

$$\Delta f_n = f_n - f_{n-1}, \quad \Delta^0 f_n = f_n,$$

$$\Delta^k f_n = \Delta \Delta^{k-1} f_n = \Delta(\Delta^{k-1} f_n), \quad (k = 1,2,\ldots);$$

and for $k > 0$,

$$\Delta^k f_n = \sum_{j=0}^{\infty} A_j^{-k-1} f_{j+n},$$

provided this series is convergent; and

A sequence $\{f_n\}$ is said to be convex if

$$\Delta^2 f_n \geq 0, \quad n = 1,2,\ldots.$$

0.8. BOUNDED VARIATION:

By '$\{t_n\} \in BV$', we mean that the sequence $\{t_n\}$ is of bounded variation, that is to say,

$$\Sigma \, |t_n - t_{n-1}| \leq K,$$

which is the same as:

$$\Sigma \, |\bar{\Delta} \; t_n| \leq K.$$

By '$\{t_n\} \in BV^k$' $k \geq 1$, we mean

$$\Sigma \, n^{k-1} \, |\bar{\Delta} \; t_n|^k \leq K.$$

Thus '$\{t_n\} \in BV^1$' is the same as $\{t_n\} \in BV$.

By '$f(x) \in BV(h,k)$' we mean that $f(x)$ is a function of bounded variation in the interval (h,k), that is,

$$\int_h^k |f'(x)|dx \leq K,$$

and by '$f(x) \in BV^p(h,k)$', for $p \geq 1$, we mean that

$$\int_h^k \{w(x)\}^{p-1} |f'(x)|^p \, dx \leq K,$$

where $w(x)$ is a suitable function of x, so that '$f(x) \in BV^1(h,k)$ is the same as '$f(x) \in BV(h,k)$'.

0.9. <u>CLASSES OF SUMMABILITY FACTORS</u>:

By '$\{\epsilon_n\} \in (P,Q)$' means 'the sequence of factors $\{\epsilon_n\}$ belongs to the class (P,Q)', that is, the series $\Sigma a_n \epsilon_n$ is summable Q whenever Σa_n is summable P. For example, $\{\epsilon_n\} \in [|c,\alpha|, |c,\beta|_k]$ if $\Sigma \epsilon_n a_n$ is summable $|c,\beta|_k$ whenever Σa_n is summable $|c,\alpha|$. This notation is due to Schur [70] and is in conformity with the terminology and notation used for Fourier-factors (Zygmund [77] p.100).

CHAPTER 1

CHAPTER I

INTRODUCTION

1.1. With the appearance of Cauchy's Analyse Algebrique in 1821, (see [25]), and Abel's researches on binominal series in 1826, (see [1]), the old hazy notion of convergence of infinite series was put on sound foundation. It was, however noticed that there were certain non-convergent series which, particularly in Dynamical Astronomy, furnished nearly correct results. After persistant efforts in which a number of celebrated leading mathematicians took part, it was only in the closing decade of the last century and in the early years of the present century that satisfactory methods were deviced so as to associate with them by processes closely connected with Cauchy's concept of convergence, certain values which may be called their 'sums' in a reasonable way. Such processes of summation of series which were formerly tabooed being divergent, have given rise to the modern regorous theory of SUMMABILITY. For the pioneering researches that, led to this theory, the credit goes 'inter-alia' to Hölder, Cesàro, Riemann, Hausdorff, Borel and others, (see Hardy [34]).

Summability being the natural generalization of 'convergence', the analogous idea of ABSOLUTE SUMMABILITY emerged as a generalization of the concept of 'absolute convergence'. In 1911, Fekete [31] introduced the absolute Cesàro summability and since then quite good contributions have been made in this direction.

In 1957, Flett [32] extended the concept of absolute summability to that of 'Generalized Absolute Summability' or 'Absolute Summability with indices, which was followed up by Mazhar [44] and others.

In the present dissertation we are concerned with a survey of some research done in this extended concept (mentioned above).

1.2. Let $\Sigma\ a_n$ be a given infinite series with sequence $\{s_n\}$ of its partial sums.

Generally, all commonly used processes of summability belong either one or the other of two kinds of processes, viz.,

(i) the T-processes,

(ii) the ϕ-processes.

A T-process is based upon the formation of a sequence of auxiliary means defined by the sequence-to-sequence transformation:

(1.2.1) $$t_n = \Sigma\ a_{n,k}\ s_k \qquad (n = 0,1,2,\ldots),$$

or, by series-to-sequence transformation:

(1.2.2) $$t_n = \sum_{k} \bar{a}_{n,k}\ a_k \qquad (n = 0,1,2,\ldots),$$

where $\bar{a}_{nk} = \sum_{j=0}^{k} a_{n,j}$,

$a_{n,k}$ being the elements of the nth row and k-th column of the Toeplitz matrix $T = (a_{nk})$.

Other types of transformations under this category are the series-to-series transformations, and the sequence-to-series transformations, with which we are not concerned here.

A ϕ-process is based upon the formation of the functional transformation defined by the sequence-to-function transformation

(1.2.3) $$t(x) = \Sigma\ \phi_n(x)\ s_n,$$

or, by the series-to-function transformation:

(1.2.4) $$t(x) = \Sigma\, \Phi_n(x)\; a_n,$$

$$\Phi_n(x) = \sum_{k=o}^{n} \phi_k(x);$$

or, by the function-to-function transformation:

(1.2.5) $$t(x) = \int \phi(x,y)\; s(y)\; dy,$$

where x is a continuous parameter, $\phi_n(x)$, or $\Phi_n(x)$ (or $\phi(x,y)$) is defined over an appropriate interval of x (or of x and y).

The series $\Sigma\, a_n$, or the sequence $\{s_n\}$, is said to be summable to a finite number s by a T-process (or summable (T) or a ϕ-process (or summable (ϕ), according as the sequence $\{t_n\}$, or the function $t(x)$, tends to s, as n tends to infinity, or as x tends to an appropriate limit, depending upon the method, (see Knopp [41], p. 474).

The series $\Sigma\, a_n$ is said to be absolutely convergent, if $\Sigma\, |a_n| < \infty$, that is,

(1.2.6) $$\Sigma\, |s_n - s_{n-1}| < \infty.$$

Thus, the absolute convergence of $\Sigma\ a_n$ may be defined as the bounded variation of $\{s_n\}$. Symbolically, we write the relation (1.2.5) as:

$$\{s_n\} \in BV.$$

Of course, absolute convergence implies convergence.

In analogy with concept of 'absolute convergence', the series $\Sigma\ a_n$, or the sequence $\{s_n\}$, is said to be absolutely summable by a T-process, or summable $|T|$, if

$$\{t_n\} \in BV.$$

Absolute summability by a ϕ-process (or the process $|\phi|$) is similarly defined, with the obvious difference that, in this case $t(x) \in BV(A, \rho)$ where (A, ρ) is a suitable interval of variation of the continuous variable x.

Extending the defintion of the process $|T|$ to absolute summability with indices, we say that the series $\Sigma\ a_n$, or the sequence $\{s_n\}$, is said to be absolutely summable by a T-process with index k, or simply summable $|T|_k$, $k \geq 1$, if $\{t_n\} \in BV^k$, that is,

$$(1.2.7) \qquad \Sigma\, n^{k-1} \left| t_n - t_{n-1} \right|^k < \infty .$$

$|T|_1$ is the same as $|T|$. Similarly $\Sigma\, a_n$, or the sequence $\left\{ s_n \right\}$, is said to be summable $|\phi|_k$, $k \geq 1$, if

$$(1.2.8) \qquad \int_A^{\rho} \left\{ w(x) \right\}^{k-1} \left| \frac{d}{dx} \left\{ t(x) \right\} \right|^k dx < \infty ;$$

where $w(x)$ depends upon the particular process, and (A, ρ) is a suitable interval of variation of x as before. Of course $|\phi|_1$ is the same as $|\phi|$.

1.3. A method of summability P is said to be conservative, briefly, P is K, if $(C,0) \subseteq P$, i.e. the convergence of any series implies its summability P and is said to be regular, briefly P is T, if P is K and also preserves sums of convergent series. The method P is said to be absolutely conservative, briefly, P is AK, if $|C,0| \subseteq |P|$, i.e., the absolute convergence of any infinite series implies its summability $|P|$, and is said to be absolutely regular, briefly, P is AT, if (i) P is AK and (ii) P is T. It has been observed by Miss Morley that a method may be AK without being

We mention in passing that regular matrix methods can not take care of even all bounded sequences, since, as proved by Steinhaus (see [29]), given any regular matrix method M, there exists a bounded sequence which is not summable M. By analogy, one might be led to ask whether it is true that no absolutely regular matrix method can sum all conditionally convergent series, (see [61]).

Necessary and sufficient conditions that a matrix method be AK were first obtained by Miss Mears [49] in 1937, and functional analytic proofs of equivalent results were given later on by Knopp and Lorentz [41], and Sunouchi [71] in 1949. Similarly, we have necessary and sufficient conditions that a ϕ-process is K (Hardy [34]), and AK(Ahmad [2], [4]; see also Das [30]).

Let P and Q be any two methods of summability (ordinary, absolute, or absolute with index). In, the case in which $P \subseteq Q$, but $Q \subseteq P$ is false, that is $P \subset Q$, the following questions can be raised:

(i) Would it be possible in some manner to restrict the order of magnitude of the terms of the series $\Sigma\, a_n$ so that, for it $Q \subseteq P$ (and in effect $P \sim Q$) ?

(ii) Would there be sequences $\{\epsilon_n\}$ such that $\Sigma\, a_n \epsilon_n$ is summable P whenever $\Sigma\, a_n$ is summable Q ?

The result answering the first question in the affirmativ are called 'Tauberian'. A result of the type $P \subset Q$ are called 'Abelian'. The sequences like $\{\epsilon_n\}$ that are required to answer the second question in the affirmative are called as summability factors(or absolute summability factors, or factors for absolute summability with index).

In the present dissertation we propose to survey the work done on the above problems in respect of absolute summability with index.

In the sequel, presenting definitions and notations of the summability methods that are involved in the present work, the author proposes to give a brief resume of the important results concerning these methods (with special reference to Absolute summability methods with index) which are necessary for discussions in the following Chapters.

1.4. SOME SPECIAL (T), $|T|$ AND $|T|_k$-PROCESSES:

I. The Cesaro methods (C), $|C|$ and $|C|_k$: In the special case in which

$$a_{n,k} = \begin{cases} A_{n-k}^{\alpha-1}, & n \geq k; \\ 0, & \text{otherwise}, \end{cases}$$

the transformation (1.2.1) reduces to s_n^{α}, the nth Cesàro-mean of order $\alpha(\alpha > -1)$, and then the corresponding (T), $|T|$ and $|T|_k$ methods are denoted by: (C,α), $|C,\alpha|$ and $|C,\alpha|_k$, $\alpha > -1$, $k \geq 1$, respectively.

In 1957, Flett [32] defined the absolute Cesàro method of order α and index k, or $|C,\alpha|_k$, $\alpha > -1$, $k \geq 1$. He studied this method in details and established the consistency theorem: for any $\alpha > -1$, $|C,\alpha|_k \subseteq |C,\beta|_r$, whenever $r \geq k \geq 1$, and $\beta \geq \alpha + \frac{1}{k} - \frac{1}{r}$. If $k = 1$, the result holds when $\alpha > -1$, $\beta > \alpha + \frac{1}{k} - \frac{1}{r}$. This extends the consistency theorem of Kogbetliantz [42]: $|C,\alpha| \subset |C,\beta|$ for $-1 < \alpha < \beta$. On the other hand, with the help of negative examples, Flett [32] demonstrated that: $|C,\beta|_r \not\subseteq |C,\alpha|_k$, where $k < r$, for any $\alpha > -1$, and $|C,\alpha|_k \not\subseteq |C,\beta|_r$, whenever $k < r$, and $\beta < \alpha + \frac{1}{k} - \frac{1}{r}$. From these results it follows that the summability methods $|C,\alpha|$ and $|C,\alpha|_k$, $k > 1$, are independent of each other. Using $|C,\alpha|_k$ as a Tauberian condition, he also proved the result: If $k > 1$, $\alpha > -1/k$, $\beta > \alpha - \frac{1}{k}$, and if $\Sigma\, a_n$ is summable $|C,\alpha|_k$, then $\Sigma\, a_n$ is summable (C,β) whenever it is summable (A).

Mehdi ([50], [51]) and others (see e.g., Mazhar [46],[47]) established some $|C,\alpha|_k$ - summability factor theorems which we propose to discuss in Chapter II.

II. <u>The methods</u> $(\overline{N},p_n)$, $|\overline{N},p_n|$ <u>and</u> $|\overline{N},p_n|_k$.

When

$$a_{n,k} = \begin{cases} p_k/P_n, & k \leq n, \\ 0, & k > n, \end{cases}$$

where $\{p_n\}$ is a sequence of constants, real or complex, such that

$$P_n = p_0 + p_1 + \dots + p_n \longrightarrow \infty, \quad \text{as} \quad n \longrightarrow \infty,$$

(1.2.1) reduces to $(\overline{N},p_n)$-mean. Then the corresponding (T), $|T|$ and $|T|_k$ -methods will be denoted by $(\overline{N},p_n)$, $|\overline{N},p_n|$ and $|\overline{N},p_n|_k$, $k \geq 1$, respectively, $|\overline{N},p_n|_1$ being the same as $|\overline{N},p_n|$.

The method $(\overline{N},p_n)$ is both T ([34], p.57) and AT [71]. For regular $(\overline{N},p_n)$-method, Sunouchi [71] proved that: If $p_{n+1}/P_n > q_{n+1}/Q_n$, then $|\overline{N},p_n| \subset |\overline{N},q_n|$, while Peyerimhoff [62] proved an elegant limitation theorem: If $p_{n+1}/P_{n+1} = O\,(p_n/P_n)$ then $\Sigma \frac{P_n}{Q_n} |a_n| < \infty$.

Rizvi [69] has proved the following Tauberian theorem for $|\overline{N},p_n|_k$-summability which extends a corresponding result of Ahmad and Rahiman [9], for $|\overline{N},p_n|$: For Σa_n to be summable $|C,0|_k$, (or $\{s_n\} \in BV^k$), $k \geq 1$, whenever it is summable $|\overline{N},p_n|_k$, $k \geq 1$, it is necessary and sufficient that, $\{t_n\} \in BV^k$, where

$$t_n = \frac{1}{P_n} \sum_{v=1}^{n} P_{v-1}\, a_v\,, \quad t_o = 0.$$

III. <u>The methods</u> (N,p_n), $|N,p_n|$ <u>and</u> $|N,p_n|_k$.

In the specialcase in which

$$a_{n,k} = \begin{cases} p_{n-k}/P_n, & n \geq k, \\ 0, & \text{otherwise,} \end{cases}$$

where $\{p_n\}$ is a sequence of constants, real or complex, and

$$P_n = p_o + p_1 + \dots + p_n \neq 0,\ P_{-1} = p_{-1} = 0,$$

the transformation (1.2.1) reduces to Norlund mean (Norlund [58 see also Woronoi [76]), or (N,p_n)-mean of the sequences $\{A_n\}$, generated by the sequence of coefficients $\{p_n\}$. Then the methods (T), $|T|$, $|T|_k$ reduces to Norlund methods (N,p_n), $|N,p_n|$ and $|N,p_n|_k$, $k \geq 1$; $|N,p_n|_1$ being the same as $|N,p_n|$.

In the special case in which

$$p_n = A_n^{\alpha-1}, \qquad \text{for} \quad \alpha > -1,$$

the corresponding Norlund mean reduces to the familiar (C,α)-mean and the summability methods (N,p_n), $|N,p_n|$ and $|N,p_n|_k$ to (C,α), $|C,\alpha|$ and $|C,\alpha|_k$, respectively. On the other hand, if $p_n = (n+1)^{-1}$, the Norlund methods reduce to corresponding Harmonic methods.

Norlund summability, though originally initiated in 1902 by Woronoi [76] and having remained unknown till pointed out by Tamarkin in 1932 [There is an amotated English translation by Tamarkin, (see [76]), of a paper by Woronoi published in the Proc. of the Eleventh Congress of Russian Naturalists and Scientists (Russian)], was independently introduced by Norlund [58] in 1919. In 1937 Mears [49] developed the concept of absolute Nörlund summability, which was later on extended to $|N,p_n|_k$-summability by Borwein and Cass [22] in 1968.

Necessary and sufficient conditions for the regularity of the Nörlund mean are:

$$p_n = o\,(|P_n|), \quad \text{as} \quad n \longrightarrow \infty$$

$$\sum_{r=o}^{n} |p_\gamma| = O(|P_n|), \quad \text{as} \quad n \longrightarrow \infty.$$

It may be noted that the Nörlund method is not necessarily absolutely regular for all type of sequences $\{p_n\}$. However, conditions for absolute regularity can be deduced from a theorem of Mears on matrix summability:

Necessary and sufficient conditions, as stated by Peyerunhoff [62], for the absolute regularity of the method $|N,p_n|$ method are:

$$\text{(i)} \qquad \lim_{n \longrightarrow \infty} \frac{p_n}{P_n} = 0, \qquad \text{and}$$

$$\text{(ii)} \qquad \sum_{n=v}^{\infty} \left|\frac{P_{n-v}}{P_n} - \frac{P_{n-1-v}}{P_{n-1}}\right| < \infty,$$

for all positive integral values of v.

The problem of inclusion for summability $|N,p_n|$ and $|N,q_n|$ was discussed by McFadden and Peyerimhoff [62]. Some of their results are of necessary and sufficient type.

Borwein and Cass [22] applied $|N,p_n|_k$-summability in some other context.

1.5. SPECIAL (ϕ), $|\phi|$ AND $|\phi|_k$ - PROCESSES:

I. The methods (J,p_n), $|J,p_n|$ and $|J,p_n|_k$.

In the special case in which

$$\phi_n = \frac{p_n x^n}{\Sigma p_n x^n}, \quad 0 \leq x < 1,$$

the transformation $t(x)$ of (1.2.2) reduces to the (J,p_n)-transform, $J(x)$, defined by:

$$J(x) \equiv J_s(x) = (\Sigma p_n x^n)^{-1} \Sigma p_n s_n x^n. \tag{1.5.1}$$

If the series

$$p(x) = \Sigma p_n x^n \tag{1.5.2}$$

is convergent for $0 \leq x < 1$, and

$$\lim_{x \to 1-0} J_s(x) = s,$$

the series Σa_n is said to be summable (J,p_n) to s, where s is finite and it is said to be summable $|J,p_n|$ if the series (1.5.2) is convergent for $0 \leq x < 1$ and if, for $c > 0$, $J(x) \in BV(c,1)$, that is,

$$\int_c^1 |J'(x)| \, dx < \infty,$$

(Ahmad [2], Chapter VIII; see also Ahmad [4] and Das [30]).

The series Σa_n, or the sequences $\{s_n\}$, is said to be summable $|J,p_n|_k$, $k \geq 1$, if the series (1.5.2) is convergent for $0 \leq x < 1$, and $J(x) \in BV^k(c,1)$ [taking $w(x) = (1-x)$ in (1.2.8)], that is, if

$$\int_c^1 (1-x)^{k-1} |J'(x)|^k \, dx < \infty, \quad 0 < c < 1,$$

(Ahmad and Rahiman [7]; see also Rahiman [68]).

It is clear that $|J,p_n|_1$ is the same as the summability $|J,p_n|$. For $k > 1$, the summability methods $|J,p_n|$ and $|J,p_n|_k$ are independent of each other (see Mazhar [45]).

The special cases of (J,p_n), $|J,p_n|$ and $|J,p_n|_k$ are:

II. <u>The Abel methods</u> (A), $|A|$, $|A|_k$, $k \geq 1$, <u>for</u> $p_n = 1$, for all n.

III. <u>The Abel methods</u> (A_λ), $|A_\lambda|$ <u>and</u> $|A_\lambda|_k$, <u>when</u> p_n <u>is given by</u>.

$$(1-x)^{-\lambda-1} = \sum_{n=o}^{\infty} p_n x^n, \text{ for } \lambda > -1, \; (|x| < 1),$$

that is, when

$$p_n = A_n^{\lambda-1}, \quad \lambda > -1.$$

IV. <u>The logarithmic methods</u> (L), $|L|$ <u>and</u> $|L|_k$. when p_n is given by

$$x^{-1} \log[(1-x)^{-1}] = \sum_{n=0}^{\infty} p_n x^n,$$

that is,

$$p_n = (n+1)^{-1}.$$

The methods (Ao), $|Ao|$ and $|Ao|_k$ are nothing but the Abel methods (A), $|A|$ and $|A|_k$ respectively.

Like $|C,\alpha|$-method, $|A|$-method is the most fundamental.

By definition it is evident that $|A| \subset (A)$. Analogous to Abel's classical theorem, we also have the result that $|C,0| \subseteq |A|$, [75]. Fekete [31] generalized this and proved that: $|C,\alpha| \subseteq |A|$, however large $\alpha(> 0)$ may be, and also showed by means of a negative example that $|A| \nsubseteq (C,\alpha)$, and hence $|A| \nsubseteq |C,\alpha|$, however large $\alpha(> 0)$ may be. This was also independently verified for Fourier series by Randels [66]. It has been demonstrated by Pati [60] that, for the conjugate series of a Fourier series, summability $|A|$ at a point, even when combined with everywhere convergence, does not necessarily imply summability $|C,1|$ at that point.

On the discovery of the fact that Dini's convergence criterion for a Fourier series at point is sufficient to ensure its summability $|A|$, Whittaker [75] was led to the consideration of the interrelation between summability (C,0), that is, convergence and summability $|A|$. Using an example suggested by Littlewood he proved that $(C,0) \subseteq |A|$. Prasad [63], on the other hand proved that $|A| \nsubseteq (C,0)$. Hyslop [37] has proved that: If, for a series Σa_n, $\Sigma \Delta(n a_n)$ is summable $|C, \alpha+1|$, that $|A| \subseteq |C,\alpha|$, for $\alpha \geq 0$; in particular, if $\{n a_n\} \in BV$, then $|A| \subseteq |C,0|$.

Concerning relation between $|C,\alpha|_k$ and $|A|_k$, Flett [32] has proved that: For $k \geq 1$, $\alpha > -1$, $|C,\alpha|_k \subseteq |A|_k$. He also showed that $|A|_r \not\subseteq |A|_k$ whenever $k < r$ and also conjectured that *for $k < r$, $|A|_k \subseteq |A|_r$. Thus, $|A|_k$ and $|A|_r$ are* mutually independent. Extending the above mentioned Tauberian theorem of Hyslop [37], Mazhar [45] has proved that: For a given series Σa_n, $|A|_k \subset |C,\alpha|_k$, whenever $\Sigma \Delta(n\, n\, a_n)$ is summable $|C, \alpha+1|_k$, for any $\alpha > -1$, and $k \geq 1$.

Concerning the family of $|J,p_n|$-methods Ahmad ([2], [3], [4]) has studied a number of problems, e.g., he has proved that: (i) $|A| \subset |L|$, (ii) $|A,\lambda| \subseteq |A,\lambda+\delta|$, for $\lambda > -1$, $\delta > 0$, (iii) $|A_\alpha| \subseteq |A_\beta|$, for $\alpha > \beta \geq -1$, (iv) (J,p_n)-method is AT, whenever $\Sigma p_n = \infty$, (v) $|\bar{N},p_n| \subset |J,p_n|$; in particular, $|\ell| \subset |L|$ [logarithmic mean transforms; ℓ_n is defined by: $\ell_o = s_o$, $\ell_1 = s_1$, $\ell_n = (\log n)^{-1} (s_o + \frac{s_1}{2} + \dots + \frac{s_n}{n+1})$, for $n = 2,3,\dots$]. Ahmad and Rahiman ([9], see also Ahmad and Varshney [10]), have also proved, for suitable 'p_n', $|J,p_n| \subseteq |\bar{N},p_n|$ under some Tauberian condition and has derived from it a couple of simplified Tauberian theorems for $|J,p_n|$-method. Extending this theorem, Rizvi ([69], Chapter V), established the following theorems:

1. For Σa_n to be summable $|C,0|_k$ (or $\{s_n\} \in BV^k$), $k > 1$, whenever it is summable $|\bar{N},p_n|_k$, $k \geq 1$, it is necessary and sufficient that $\{t_n\} \in BV^k$, where

$$t_n = (P_n)^{-1} \sum_{\gamma=1}^{n} P_{\gamma-1}\, a_\gamma , \quad t_o = 0.$$

2. If, for $k \geq 1$, Σa_n is summable $|J,p_n|_k$, $\{t_n\} \in BV^k$ and if $\{p_n\}$ is such that,

(i) $\dfrac{n\, p_n}{P_{n-1}} < C,$ for $n = 1,2,\ldots,$

(ii) $\dfrac{\{M(w)\}^k}{w^{k-1}} > C,$ for $w \geq 1,$

(iii) $\left\{ \sum_{\nu=o}^{\infty} P_\nu\, e^{-\nu/3w} \Big/ \sum_{\nu=o}^{\infty} P_\nu\, e^{-\nu/w} \right\}^{k-1}$ is bounded,

for $w \geq 1$, then series Σa_n is summable $|\bar{N},p_n|_k$, where

$$M(w) = \frac{\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} \sum_{\nu=n}^{\infty} e^{-\nu/w} \sum_{\mu=n}^{\nu} (2\mu-\nu)\, p_\mu\, P_{\nu-\mu}}{\sum_{\nu=o}^{\infty} e^{-\nu/w} \sum_{\mu=o}^{\nu} (\nu-\mu+1)\, p_\mu\, p_{\nu-\mu}},$$

and C is a constant, not necessarily the same at each occurrence.

3. *If* Σa_n *is summable* $|J, p_n|_k$, *and* $\{t_n\} \in BV^k$, and if $\{p_n\}$ satisfies the conditions as in the preceding theorem, then Σa_n is summable $|C,0|_k$.

4. If Σa_n is summable $|J, p_n|_k$, and $\left\{ a_n \frac{P_{n-1}}{p_n} \right\} \in BV^k$, and if $\{p_n\}$ satisfies the same conditions as in the preceding theorem, and if, in addition:

(iv) $\left(\frac{1}{P_n} \sum_{\nu=1}^{n} \frac{P_\nu}{\nu}\right)^{k-1}$ is bounded for $k \geq 1$, then Σa_n is summable $|C,0|_k$.

1.6. <u>APPLICATIONS TO FOURIER SERIES</u>:

Let $f(t)$ be a periodic function with period 2π, and integrable in the sense ofLebesgue over $(-\pi, \pi)$. The Fourier series of $f(t)$ is given by:

$$(1.6.1) \qquad f(t) \sim \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$\equiv \sum_{n=o}^{\infty} A_n(t),$$

where

$$a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt;$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos\, nt\, dt, \quad n = 1,2,\ldots;$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin\, nt\, dt, \quad n = 1,2,\ldots .$$

Localization Problem: The well known Principle of localization [73] states that the behaviour of Fourier series regarding its convergence for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of the point only. In other words, however small δ may be, the behaviour of $s_n(x)$ (the nth partial sum of a Fourier series) depends on the nature of the generating function $f(t)$ in the interval $(x-\delta, x+\delta)$ only, and is not affected by the values which it takes outside the interval. In 1933, Prasad [63] proved that the summability $|A|$ of a Fourier series at point depends only upon the behaviour of the generating function in the immediate neighbourhood of the point at which the absolute summability of the Fourier series is considered. Later on this line has been followed upon a number of workers for various absolute summability methods e.g., Bosanquet [23] for $|C,\beta|$,

$\beta > 1$; Bhatt [13], for $|N,p_n|$. On the other hand Bosanquet and Kestelman [24] and Randels [66] have proved that the summability $|C,1|$ of a Fourier series at a point is not a local property; this line was also followed up by many other workers. Thus, it follows from the consistency theorem of $|C|$-summability that the summability $|C,\alpha|$, $-1 < \alpha \leq 1$, is not a local property. Then the problem arises as to what should be the nature of the generating function $f(x)$ of the Fourier series so that its summability $|C,\alpha|$, $\alpha \geq 1$, may become a local property. The answer to this question was also given by various authors, e.g. Mohanty [55], for $|C,1|$-summability, Bhatt [11] and Jurkat and Peyerimhoff [39] for $|C,\alpha|$; also Bhatt [13] for $|N,p_n|$, etc.

This problem has been studied by Bor [21] for $|\bar{N},p_n|_k$-summability of Factored Fourier which we propose to discuss in Chapter III.

Recently Ahmad [6] has discussed the applications of absolute summability to Fourier series, its conjugate series and their derived series, by methods based on power series, including the those of the methods $|J,p_n|_k$. In the last Chapter of this dissertation we propose to discuss the $|J,p_n|_k$-summability of Fourier, as studied by Ahmad and Rahiman ([7], [8]; see also Rahiman [68], Chapter VI).

We have to further point out the Chapter II also contains some problem on $|C,\alpha|_k$-summability factors of Fourier series, while in Chapter IV, results concerning $|N,p_n|_k$-summability factors of power series and Fourier series.

CHAPTER 2

CHAPTER II

SUMMABILITY FACTORS OF A FOURIER SERIES BY ABSOLUTE SUMMABILITY WITH INDEX

2.1. DEFINITIONS AND NOTATIONS:

Let Σa_n be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{u_n^\alpha\}$ and $\{t_n^\alpha\}$, denote the nth Cesàro mean of order $\alpha(\alpha > -1)$ of the sequences $\{s_n\}$ and $\{n a_n\}$ respectively.

The series Σa_n is said to be summable $|C,\alpha|$, if

$$(2.1.1) \qquad \sum_{n=1}^{\infty} |u_n^\alpha - u_{n-1}^\alpha| < \infty, \quad \text{(Fekete [31])}.$$

The series Σa_n is said to be summable $|C,\alpha|_k$, $k \geq 1$, if

$$(2.1.2) \qquad \sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha| < \infty, \quad \text{(Flett [32])}.$$

But, since

$$(2.1.3) \qquad t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha) \quad \text{(Kogbetliantz [42])},$$

condition (2.1.2) can be replaced as:

$$(2.1.4) \qquad \sum_{n=1}^{\infty} \frac{1}{n} |t_n^{\alpha}|^k < \infty .$$

A series $\Sigma\, a_n$ is said to be summable $|A|$ if the series $\Sigma\, a_n\, x^n$ is convergent for $0 \leq x < 1$ and its sum function $\phi(x)$ satisfies the condition

$$(2.1.5) \qquad \int_0^1 |\phi'(x)|\, dx < \infty, \quad \text{(Whittakar [75])}.$$

A series $\Sigma\, a_n$ is said to be summable $|A|_p$, $p \geq 1$ if the series $\Sigma\, a_n\, x^n$ is convergent for $0 \leq x < 1$, and its sum function $\phi(x)$ satisfies the condition

$$(2.1.6) \qquad \int_0^1 (1-x)^{p-1}\, |\phi'(x)|^p\, dx < \infty, \quad \text{(Flett [32])}.$$

Let $\{p_n\}$ be a sequence of positive numbers, such that

$$P_n = p_o + p_1 + \dots + p_n \longrightarrow \infty.$$

The series $\Sigma\, a_n$ is said to be summable $|\overline{N}, p_n|$, if $t_n \in BV$, where

$$(2.1.7) \qquad t_n = \frac{1}{P_n} \sum_{k=o}^{n} p_k s_k .$$

Let $f(t)$ be a periodic function with period 2π, and integrable in the sense of Lebesgue over $(-\pi,\pi)$. The Fourier series of $f(t)$ is given by:

$$(2.1.8) \qquad f(t) \sim \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

$$\equiv \sum_{n=0}^{\infty} A_n(t).$$

Then, for $n \geq 1$

$$(2.1.9) \qquad \pi A_n(x) = \int_0^{\pi} \phi(t) \cos nt \, dt,$$

where $\phi(t) = f(x+t) + f(x-t) - 2f(x)$.

We write

$$(2.1.10) \qquad \psi(t) = \int_t^{\gamma} u^{-1} |\phi(u)| du, \quad 0 < \gamma \leq \pi$$

$$(2.1.11) \qquad \mu_n = (\prod \log^{\gamma} n) \, (\log^{\ell} n)^{1+\epsilon}, \ \log n_o > 0, \epsilon > 0, \ell \geq 2,$$

where $\log^{\ell} n = \log(\log^{\ell-1} n)$, ..., $\log^2 n = \log\log n$.

2.2. INTRODUCTION:

Whittakar [75] in 1930, proved that the series

$$\sum_{n=1}^{\infty} \frac{A_n(x)}{n^{\alpha}}, \quad \alpha > 0,$$

is summable $|A|$ almost everywhere.

Later, Prasad [64] demonstrated that the series

$$\sum_{n=n_0}^{\infty} \frac{A_n(x)}{\mu_n}$$

is summable $|A|$ almost everywhere.

Chow [28], on the other hand, has shown that the series $\Sigma\ \lambda_n A_n(x)$ is summable $|C,1|$ almost everywhere, provided $\{\lambda_n\}$ is a convex sequence satisfying the condition

$$\Sigma\ n^{-1} \lambda_n < \infty .$$

Cheng [26] in 1948, established the following result.

<u>Theorem 2.2.1 [26]</u>. If

$$\Phi(t) \equiv \int_0^t \phi(u)\, du = o(t)$$

as $t \longrightarrow 0$, then the series $\sum_{n=2}^{\infty} A_n(x)/(\log n)^{1+\delta}$, $\delta > 0$ is summable $|C,\alpha|$, $\alpha > 1$.

Mehdi [51] obtained necessary and sufficient conditions to be satisfied by a sequence (ϵ_n) of real or complex numbers in the form of following results.

Theorem 2.2.2 [51]. Let $1 \leq k < +\infty$ and $\alpha \geq 0$. The necessary and sufficient condition for $\Sigma\, \epsilon_n a_n$ to be summable $|A|_k$ whenever $\Sigma\, a_n$ is summable $|C,\alpha|$, are:

(i) $\quad \Delta^{\alpha} \epsilon_n = O(n^{-\alpha})$;

(ii) $\quad \epsilon_n = O(1)$.

Theorem 2.2.3 [51]. Let $1 < k \leq +\infty$, $\beta \geq 0$, and let α be a non-negative integer. The necessary and sufficient conditions for $\Sigma\, \epsilon_n a_n$ to be summable $|C,\beta|_k$ whenever $\Sigma\, a_n$ is summable $|C,\alpha|$, are:

(ia) $\quad \Delta^{\alpha} \epsilon_n = O(n^{-\alpha}) \qquad (\beta > 1 - \frac{1}{k})$

(ib) $\quad \Delta^{\alpha} \epsilon_n = O[n^{-\alpha} (\log n)^{-\frac{1}{k}}] \qquad (\beta = 1 - 1/k)$

and (iia) $\varepsilon_n = O[n^{\beta-\alpha-1+1/k}]$ $(\beta < \alpha + 1 - \frac{1}{k})$

(iib) $\varepsilon_n = O[(\log n)^{-1/k}]$ $(\beta = \alpha + 1 - \frac{1}{k})$

(iic) $\varepsilon_n = O(1)$ $(\beta > \alpha + 1 - \frac{1}{k})$.

when $0 \leq \beta < 1 - 1/k$, the required condition is only (iia).

<u>Remarks</u>: (1) When $0 \leq \beta < 1 - 1/k$ condition (iia) implies (ia) so that (ia) may be omitted.

(2) If $k = +\infty$, the conditions are free of log n and may be stated more simply as (ia) and

(iia)$'$ $\varepsilon_n = O(n^{\beta-\alpha-1})$ $(\beta \leq \alpha + 1)$

(iic)$'$ $\varepsilon_n = O(1)$ $(\beta \geq \alpha+1)$.

(3) If $1 < k < +\infty$ and β is also an integer then the conditions are again free of log n and becomes more simply (ia), (iia), and (iic). This is the only known instance of summability factors for which the form of the conditions for general α and β is not the same as the form of the conditions for integral α and β.

It is known that the summability $|\bar{N},p_n|$ and summability $|C,\alpha|_k$ are, in general, independent of each other. Then, (1) It is, therefore, natural to find out suitable summability factor $\{\epsilon_n\}$, so that $\Sigma\, a_n\epsilon_n$ may be summable $|C,\alpha|_k$, $\alpha > -1$, $k \geq 1$, whenever $\Sigma\, a_n$ is summable $|\bar{N},p_n|$, and (2) If $\Sigma\, a_n$ is summable $|C,\alpha|_k$ then $\Sigma\, a_n\epsilon_n$ may be summable $|\bar{N},p_n|$. Mazhar ([46], [47]) has examined the summability factor problem of the both types and we will discuss them here respectively.

Mazhar [47] first gave the following result:

Theorem 2.2.4 [47]. Let $\Sigma\, a_n$ be summable $|\bar{N},p_n|$. Then, $\Sigma\, a_n\epsilon_n$ is summable $|C,\beta|_k$, $\beta > -1$, $k > 1$, if

(i) (a) $\quad \Delta\epsilon_n = O(p_n/P_n), \quad \beta > 1 - \frac{1}{k}$;

(i) (b) $\quad \Delta\epsilon_n = O(\frac{p_n}{P_n}(\log n)^{-1/k}), \quad \beta = 1 - \frac{1}{k}$;

(ii)(a) $\quad \epsilon_n = O(\frac{p_n}{P_n} n^{\beta+1/k-1}), \quad \beta < 2 - \frac{1}{k}$;

(ii)(b) $\quad \epsilon_n = O(\frac{np_n}{P_n}(\log n)^{-1/k}), \quad \beta = 2 - \frac{1}{k}$;

(ii)(c) $\quad \epsilon_n = O(\frac{np_n}{P_n}), \quad \beta > 2 - \frac{1}{k}$;

where $p_{n+1} = O(p_n)$ and $n\, p_n = O(P_n)$.

If $-1 < \beta < 1 - \frac{1}{k}$, then the required condition is (ii)(a

If $-1 < \beta \leq 2 - \frac{1}{k}$, then the conditions (i)(a), (i)(b), (ii)(a) and (ii)(b) are necessary and sufficient for the validit of the theorem. On the other hand, when $\beta > 2 - \frac{1}{k}$, the conditions (i)(a) and (ii)(c) are necessary and sufficient provided $P_n = O(n\, p_n)$.

<u>Remarks</u>: I. If β is an integer and $k > 1$, then the terms involving $\log n$ will disappear and our conditions become (i)(a), (ii)(b) and (ii)(c).

II. For $k = 1$, in view of lemma 2 (Mehdi [51]) conditions are

(i)(a) $\Delta\epsilon_n = O(\frac{p_n}{P_n})$, $\beta > 0$;

(ii)(a) $\epsilon_n = O(\frac{p_n}{P_n} n^{\beta})$, $\beta \leq 1$;

(ii)(c) $\epsilon_n = O(\frac{n\, p_n}{P_n})$, $\beta > 1$.

when $-1 < \beta \leq 0$, the required condition is (ii)(a).

III. For $p_n = 1$, we get theorem 2.2.3 (Mehdi [51]) for the case $\alpha = 1$.

Again, Mazhar [46] obtained the following result for the summability factor of second type as follows:

Theorem 2.2.5 [46]. The necessary and sufficient conditions for the series $\Sigma\, a_n \epsilon_n$ to be summable $|\bar{N},p_n|$ whenever $\Sigma\, a_n$ is summable $|C,\alpha|_k$, $\alpha \geq 0$, $k \geq 1$, are

(i) $\left\{ n^{\alpha+1-\frac{1}{k'}} \Delta^{\alpha}\left(\frac{\epsilon_n}{n}\right) \right\} \in \ell^{k'}$;

where $\frac{1}{k} + \frac{1}{k'} = 1$;

(ii)(a) $\left\{ n^{-\frac{1}{k'}} \epsilon_n \right\} \in \ell^{k'}$, $0 \leq \alpha \leq 1$;

(ii)(b) $\left\{ n^{\alpha-\frac{1}{k'}} \left(\frac{p_n}{P_n}\right) \epsilon_n \right\} \in \ell^{k'}$, $\alpha > 1$;

where (a) $p_n = O(p_{n+1})$, (b) $(n+1)\, p_n = O(P_n)$ and (c) $P_n = O(n^{\alpha} p_n)$ $(\alpha > 1)$.

It may be remarked that this theorem includes, as a special case for $k = 1$, of the following theorem of Mohaptra [57].

Theorem 2.2.6 [57]. Let the sequence $\{p_n\}$ satisfy the following:

(2.2.1) $$p_n = O(p_{n+1});$$

(2.2.2) $$(n+1)\, p_n = O(P_n);$$

(2.2.3) $$P_n = O(p_n\, n^{\alpha}),\ \alpha > 1.$$

The necessary and sufficient conditions to be satisfied by a sequence $\{\epsilon_n\}$, such that $\Sigma\, a_n \epsilon_n$ is summable $|\bar{N}, p_n|$, whenever $\Sigma\, a_n$ is summable $|C,\alpha|$, $\alpha \geq 0$ are

(2.2.4) $$\epsilon_n = \begin{cases} O(1), & 0 \leq \alpha \leq 1, \\ O\left(\dfrac{P_n\, n^{-\alpha}}{p_n}\right), & \alpha > 1 \end{cases}$$

(2.2.5) $$\Delta^{\alpha}\left(\frac{\epsilon_n}{n}\right) = O(n^{-\alpha-1}).$$

On the other hand if we take $p_n = 1$, we get the following result of Mehdi [51].

Theorem 2.2.7. Let $\alpha \geq 0$, $k > 1$. The necessary and sufficient conditions for $\Sigma\, a_n \epsilon_n$ to be summable $|C,1|$ whenever $\Sigma\, a_n$ is summable $|C,\alpha|_k$ are

(i) $$\left\{ n^{\alpha+1-\frac{1}{k'}}\, \Delta^{\alpha}\left(\frac{\epsilon_n}{n}\right)\right\} \in \ell';$$

where $\frac{1}{k} + \frac{1}{k'} = 1$;

$$(ii)'(a) \qquad \sum_{n=1}^{\infty} \frac{|\epsilon_n|^{k'}}{n} < \infty, \qquad \alpha \leq 1;$$

$$(ii)'(b) \qquad \sum_{n=1}^{\infty} n^{-1+\alpha k'-k'} |\epsilon_n|^{k'} < \infty, \quad \alpha \geq 1.$$

Similarly, on taking $p_n = 1/n+1$, the result concerning $|R, \log n, 1|$ summability factors of infinite series can be obtained.

In 1970, Hsiang [36] proved the following theorems:

<u>Theorem 2.2.8 [36]</u>. If

$$\Phi(t) = o(t);$$

as $t \longrightarrow +0$, then the series $\sum_{n=1}^{\infty} A_n(x)/n^{\alpha}$ is summable $|C,1|$ for every $\alpha > 0$.

<u>Theorem 2.2.9 [36]</u>. If

$$\Phi(t) = o\left\{ t/\prod_{v=1}^{\lambda-1} \log^{v}(1/t)\right\};$$

as $t \longrightarrow +0$, then the series

$$\Sigma \; A_n(x)/(\prod_{v=1}^{\ell-1} \log^v n)\,(\log^{\ell} n)^{1+\epsilon} \; ;$$

is summable $|C,1|$ for every $\epsilon > 0$.

Pandey [59], proved the following result.

Theorem 2.2.10 [59]. If

$$\psi(t) = \int_t^{\delta} \frac{|\phi(u)|}{u}\, du = O\left\{ \log^{\ell}(\tfrac{1}{t})^{\eta} \right\} ;$$

as $t \longrightarrow +0$, $0 \leq \delta \leq \pi$, then the series $\sum_{n=0}^{\infty} A_n(x)/\mu_n$ is summable $|C,1|$ for $0 < \eta < \epsilon$.

The conditions of Theorem 2.2.10 are less stringent than those of Theorem 2.2.1, 2.2.8 and 2.2.9.

Recently, Sulaiman [72] gave the following extension of Theorem 2.2.10.

Theorem 2.2.11 [72]. Let $\{\lambda_n\}$ be any sequence of constants. Let $g(u)$ and $h(u)$ be positive functions such that $H(u) = u\, h(u)$, $u^{\beta}\, g(\frac{1}{u})$ are both non-decreasing for some β $0 < \beta < 1$. Suppose, for $k \geq 1$,

$$\psi(t) = O\left\{g\left(\tfrac{1}{t}\right)\right\}, \quad t \longrightarrow 0;$$

$$\sum_{n=1}^{\infty} n^{2k-\delta k-1} |\lambda_n|^k [h(n)]^k [g(n)]^k < \infty ;$$

and $\sum_{n=1}^{\infty} n^{2k-1} |\Delta\lambda_n|^k [h(n)]^k [g(n)]^k < \infty$.

Then, the series $\Sigma\, n\lambda_n\, h(n)\, A_n(x)$ is summable $|C,\delta|_k$, $0 < \delta \leq 1$.

We reproduce the proof as given by Sulaiman by using following lemmas:

Lemma 2.2.1 [26]. If $\sigma > -1$ and $\sigma - \delta > 0$, then

$$\sum_{n=\mu}^{\infty} \frac{A_{n-\mu}^{\delta}}{n\, A_n^{\sigma}} = \frac{1}{\mu\, A_{\mu}^{\sigma-\delta-1}} .$$

Lemma 2.2.2. The series $\Sigma\, \lambda_n\, a_n$ is summable $|C,\delta|_k$, $k \geq 1$, $0 < \delta \leq 1$, if the following hold:

$$\sum_{n=1}^{\infty} n^{k-\delta k-1} |\lambda_n|^k |s_n|^k < \infty;$$

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta\lambda_n|^k |s_n|^k < \infty .$$

where $s = n$ or u^{-1}.

Thereofre,

$$|I_1| = O\left\{ nH(n) \int_0^{n^{-1}} |\phi(u)|du \right\} = O\left\{H(n)g(n)\right\} ,$$

$$|I_2| = O\left\{ H(n) \int_{n^{-1}}^{\pi} u^{-1} |\phi(u)|du \right\} = O\left\{H(n)g(n)\right\} .$$

Hence

$$|s_n(x)| = O\left\{ H(n)g(n)\right\},$$

$$|s_n(x)|^k = O\left\{[H(n)]^k [g(n)]^k\right\}.$$

The theorem follows by making use of lemma 2.2.2.

Remark: By putting $k = 1$, $\delta = 1$, $h(n) = n^{-1}$, $\lambda_n = \mu_n$ and $g(n) = (\log^{\ell} n)^{\eta}$ in theorem 2.2.11. We obtain 2.2.10.

<u>Lemma 2.2.3.</u> Suppose $g(u)$ is a positive function such that $u^{\beta} g(1/u)$ nondecreasing for some β, $0 < \beta < 1$. If

$$\psi(t) = O(g(1/t)), \quad t \longrightarrow 0,$$

then

$$\int_0^t |\phi(u)| du = O\left\{ t\, g(1/t)\right\} .$$

<u>Proof of Theorem 2.2.11.</u>

Let $S_n(x)$ be the n-th partial sum of the sequence $\left\{H(n)\ A_n(x)\right\}$. Then, we have

$$S_n(x) = \sum_{v=1}^{n} H(v) \int_0^{\pi} \cos(vu)\ \phi(u)du$$

$$= \left\{ \int_0^{n^{-1}} + \int_{n^{-1}}^{z} \right\} \left\{ \sum_{v=1}^{n} H(v)\ \cos(uv)\quad \phi(u)du \right\}$$

$$= I_1 + I_2, \quad \text{say.}$$

Since $H(u)$ is non-negative, non-decreasing, thus, by Abel's lemma

$$\left| \sum_{v=1}^{n} H(v)\ \cos(vu) \right| = O\left\{ H(n) \max_{1 \le v \le n} \left| \sum_{v=r}^{n} \cos(vu) \right. \right\}$$

$$= O\left\{ sH(n) \right\} ,$$

CHAPTER 3

CHAPTER III

ON THE $|\bar{N},p_n|_k$-SUMMABILITY FACTORS FOR INFINITE SERIES

3.1 DEFINITIONS AND NOTATIONS:

Let $\Sigma\, a_n$ be a given infinite series with the partial sums s_n. Let p_n be a sequence of positive constants, such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty, \qquad \text{as} \quad n \longrightarrow \infty,$$

where $P_{-i} = p_{-i} = 0, \quad i \geq 1.$

The sequence-to-sequence transformation:

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence $\{t_n\}$ of the $(\bar{N},p_n)$-means of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$.

The series $\Sigma\, a_n$ is said to be summable $|\bar{N},p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \, |t_n - t_{n-1}|^k < \infty, \quad \text{(Bor [14])}.$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N},p_n|_k$ summability is the same as $|C,1|_k$ (resp. $|\bar{N},p_n|$) summability. Also, if we take $k = 1$ and $p_n = 1/(n+1)$, summability $|\bar{N},p_n|_k$ is equivalent to the summability $|R, \log n,1|$.

The series $\Sigma\ a_n$ is said to be bounded $[\bar{N},p_n]_k$, $k \geq 1$, if

$$\sum_{v=1}^{n} p_v |s_v|^k = O(P_n), \qquad \text{(Bor [15])}$$

as $n \longrightarrow \infty$.

If we take $k = 1$, then $[\bar{N},p_n]_k$, boundedness is the same as $[\bar{N},p_n]$ boundedness.

3.2 <u>INTRODUCTION</u>:

Bor [16] has proved the following result:

<u>Theorem 3.2.1 [16]</u>. Let $\Sigma\ a_n$ be bounded $[\bar{N},p_n]$. Let $\{p_n\}$ be a sequence, such that

(3.2.1) $\qquad P_n \longrightarrow \infty$;

as $n \longrightarrow \infty$, and

(3.2.2) $\qquad \frac{1}{n} = O(p_n).$

Suppose there are sequences $\{\beta_n\}$ and $\{\lambda_n\}$, such that

(3.2.3) $\qquad |\Delta\lambda_n| \leq \beta_n$;

(3.2.4) $\qquad \beta_n \longrightarrow \infty, \quad$ as $\quad n \longrightarrow \infty$;

(3.2.5) $\qquad \sum_{n=1}^{\infty} n P_n |\Delta\beta_n| < \infty$;

(3.2.6) $\qquad P_n|\lambda_n| = O(1) \quad$ as $\quad n \longrightarrow \infty.$

Then, the series $\Sigma\, a_n\lambda_n$ is summable $|\bar{N},p_n|$.

Recently, Bor [19] generalized the theorem 3.2.1, for $|\bar{N},p_n|_k$, $k \geq 1$, as follows:

<u>Theorem 3.2.2 [19]</u>. Let $\Sigma\, a_n$ be bounded $[\bar{N},p_n]_k$. If the sequences $\{P_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are such that conditions (3.2.2) - (3.2.6) of Theorem 3.2.1 are satisfied, then the series $\Sigma\, a_n\lambda_n$ is summable $|\bar{N},p_n|_k$, $k \geq 1$.

<u>Remark</u>: It should be noted that if we take $k = 1$ in Theorem 3.2.2, then we will get Theorem 3.2.1.

The following lemma is needed for the proof of Theorem 3.2.2.

Lemma 3.2.1. Under the conditions on $\{P_n\}$ and $\{\beta_n\}$ as taken in statement of the theorem, the following conditions hold, when (3.2.5) is satisfied.

$$n\, P_n\, \beta_n = O(1), \quad \text{as} \quad n \longrightarrow \infty; \tag{3.2.7}$$

$$\sum_{n=1}^{\infty} P_n \beta_n < \infty . \tag{3.2.8}$$

Proof of the Theorem 3.2.2.

Let $\{T_n\}$ be the sequence of the $(\bar{N}, p_n)$ mean of the series $\Sigma\, a_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{V=0}^{n} p_v \sum_{r=0}^{v} a_r \lambda_r = \frac{1}{P_n} \sum_{V=0}^{n} (P_n - P_{v-1}) a_v \lambda_v .$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n\, P_{n-1}} \sum_{v=1}^{n} a_v \lambda_v .$$

Using Abel's transformation, we get

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \triangle(P_{v-1}\lambda_v) s_v + \frac{1}{P_n} p_n s_n \lambda_n$$

$$= \frac{1}{P_n} p_n s_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v +$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \quad {}_v s_v$$

$$= T_{n,1} + T_{n,2} + T_{n,3}, \quad \text{say.}$$

To prove the theorem, by Minkowski's inequality, it is sufficien to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1,2,3.$$

Since $|\lambda_n| = O(1/P_n) = O(1)$, by (3.2.6), we have

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} |T_{n,1}|^k = \sum_{n=1}^{m} |\lambda_n|^k |p_n| |s_n|^k \frac{1}{P_n}$$

$$= \sum_{n=1}^{m} |\lambda_n|^{k-1} |\lambda_n| |p_n| |s_n|^k \frac{1}{P_n}$$

$$= O(1) \sum_{n=1}^{m} |\lambda_n| |p_n| |s_n|^k = O(1) \sum \triangle|\lambda_n| \sum_{r=1}^{n} p_r |s_r|^k$$

$$+ O(1) |\lambda_m| \sum_{n=1}^{m} p_n |s_n|^k$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| P_n + O(1) |\lambda_m| P_m$$

$$= O(1) \sum_{n=1}^{m-1} \beta_n P_n + O(1) |\lambda_m| P_m = O(1);$$

as $m \longrightarrow \infty$, by virtue of the hypothesis of the theorem and lemma.

Now, applying Hölder's inequality, we have

$$\sum_{n=2}^{m+n} (P_n/p_n)^{k-1} |T_{n,2}|^k = O(1), \quad \text{as } m \longrightarrow \infty.$$

Finally, using the fact that

$\frac{1}{V} = O(p_v)$, by (3.2.2), we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,3}|^k = O(1);$$

as $m \longrightarrow \infty$; by virtue of the hypothesis of the theorem and lemma.

Therefore, we get

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} |T_{n,r}|^k = O(1);$$

as $m \longrightarrow \infty$, for $r = 1,2,3$.

This completes the proof of the theorem.

3.3. LOCAL PROPERTY OF $|\bar{N},p_n|_k$-SUMMABILITY:

Let $f(t)$ be a periodic function with period 2π, Lebesgue integrable over $(-\pi,\pi)$ and we also assume that the constant term in the Fourier series of $f(t)$ is zero, that is

$$\int_{-\pi}^{\pi} f(t)dt = 0,$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \quad \sum_{n=1}^{\infty} A_n(t).$$

Mohanty [54] has demonstrated that the summability $|R, \log n,1|$ of

$$(3.3.1) \qquad \sum \frac{A_n(t)}{\log (n+1)}$$

at $t = x$ is a local property of the generating function of $\Sigma A_n(t)$.

Matsumoto [43] improved this result by replacing the series (3.3.1) by

$$\Sigma \frac{A_n(t)}{\{\log \log(n+1)\}^{1+\epsilon}} \; ; \; \epsilon > 0. \tag{3.3.2}$$

Generalizing the above result Bhatt [12] proved the following theorem:

Theorem 3.3.1 [12]. If $\{\lambda_n\}$ is a convex sequence such that $\Sigma\, n^{-1} \lambda_n$ is convergent, then the summability $|R, \log n, 1|$ of the series $\Sigma\, A_n(t) \lambda_n \log n$ at a point can be ensured by a local property.

Mishra [53] has proved Theorem 3.3.1 in the form of the following result:

Theorem 3.3.2 [53]. Let the sequence $\{p_n\}$ be such that

$$P_n = O(n\, p_n) \tag{3.3.3}$$

$$P_n \Delta p_n = O(p_n\, p_{n+1}). \tag{3.3.4}$$

Then, the summability $|\bar{N}, p_n|$ of the series

$$\sum_{n=1}^{\infty} A_n(t) \lambda_n P_n (n\, p_n)^{-1}, \tag{3.3.5}$$

where $\{\lambda_n\}$ is as in Theorem 3.3.1, at a point can be ensured by a local property.

Bor [17] has generalized Theorem 4.3.2, for $|\bar{N},p_n|_k$ summability provided that $k \geq 1$, as follows:

Theorem 3.3.3. [17]. Let $k \geq 1$ and let the sequence $\{p_n\}$ be such that condition (3.3.3) and (3.3.4) of Theorem 3.3.2 are satisfied. Then, the summability $|\bar{N},p_n|_k$ of the series (3.3.5), where $\{\lambda_n\}$ is as given in Theorem 3.3.1, at a point can be ensured by a local property.

Remark. Since $\{\lambda_n\}$ is a convex sequence therefore the sequence $\{(\lambda_n)^k\}$ is also convex and $\Sigma \frac{1}{n} (\lambda_n)^k < \infty$.

Recently, Bor [21] generalized the above result under more appropriate conditions than those given in Theorem 3.3.3 as under:

Theorem 3.3.4 [21]. Let $k \geq 1$ and let the sequences $\{p_n\}$ and $\{\lambda_n\}$ be such that

(3.3.6) $\qquad \Delta X_n = O(\frac{1}{n});$

$$(3.3.7) \qquad \sum_{n=1}^{\infty} X_n^{k-1} \frac{|\lambda_n|^k + |\lambda_{n+1}|^k}{n} < \infty ;$$

$$(3.3.8) \qquad \sum_{n=1}^{\infty} (X_n^k + 1) \, |\Delta \lambda_n| < \infty;$$

where $X_n = P_n/n\, p_n$. Then, the summability $|\overline{N},p_n|_k$ of the series

$$(3.3.9) \qquad \sum_{n=1}^{\infty} A_n(t) \lambda_n X_n ;$$

at a point can be ensured by a local property.

<u>Remark</u>. It is known (see [12]) that if $\{\lambda_n\}$ is a convex sequence and $\Sigma\, n^{-1} \lambda_n$ is convergent, then

$$\lambda_n \geq \lambda_{n+1} \geq 0, \quad \lambda_n \log n = o(1), \text{ and } \Sigma \log n \Delta \lambda_n < \infty,$$

so that (3.3.8) is a natural condition to impose.

The following lemma is needed for the proof of main theorem.

<u>Lemma 3.3.1</u>. Let $k \geq 1$ and let the sequences $\{p_n\}$ and $\{\lambda_n\}$ be such that conditions (3.3.6) - (3.3.8) of

Theorem 3.3.4 are satisfied. If $\{s_n\}$ is bounded, then the series

$$(3.3.10) \qquad \sum_{n=1}^{\infty} a_n \lambda_n X_n$$

is summable $|\bar{N},p_n|_k$.

<u>Proof</u>. Let $\{T_n\}$ denote the $(\bar{N},p_n)$ means of the series (3.3.10). Then, by definition, we have that

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum a_z \lambda_z X_z$$

$$= \frac{1}{P_n} \sum_{v=0}^{n} (P_v - P_{v-1})\, a_v \lambda_v X_v,\ X_o = 0.$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1}\, a_v \lambda_v X_v.$$

By Abel's transformation, we get

$$T_n - T_{n-1} = \frac{-p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v X_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v X_v P_v \Delta \lambda_v$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \lambda_{v+1} \Delta X_v + \frac{s_n \lambda_n\, p_n X_n}{P_n}$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}\text{, say.}$$

To complete the proof of the lemma, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k < \infty ;$$

for $r = 1,2,3,4.$

Now, applying Hölder's inequality, we have

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,1}|^k \leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} |s_v|^k |\lambda_v|^k p_v X_v^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} p_v |s_v|^k |\lambda_v|^k X_v^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} X_v^k \frac{p_v}{P_v} |\lambda_v|^k$$

$$= O(1) \sum X_v^{k-1} \frac{|\lambda_v|^k}{v}$$

$$= O(1);$$

as $m \longrightarrow \infty$, by (3.3.7) and hypothesis.

Since

$$\sum_{v=1}^{n-1} P_v|\Delta\lambda_v| \le P_{n-1} \sum_{v=1}^{n-1} |\Delta\lambda_v| \Rightarrow \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |\Delta\lambda_v|$$

$$\le \sum_{v=1}^{n-1} |\Delta\lambda_v| = O(1);$$

by (3.3.8), we have that

$$\sum_{n=1}^{m+1} (P_n/p_n)^{k-1} |T_{n,2}|^k = O(1);$$

by (3.3.8) and the hypothesis of the lemma.

Using the fact that $\Delta X_v = O(1/v)$; by (3.3.6) we have

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,3}|^k = O(1);$$

by (3.3.7) and the hypothesis of the lemma.

Finally, we have that

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} |T_{n,4}|^k = O(1) \sum_{n=1}^{m} X_n^k \frac{p_n}{P_n} |\lambda_n|^k$$

$$= O(1) \sum_{n=1}^{m} X_n^{k-1} \frac{|\lambda_n|^k}{n}$$

$$= O(1);$$

as $m \longrightarrow \infty$, by (3.3.7) and the hypothesis of the lemma.

Therefore, we get

$$\sum_{n=1}^{m} (P_n/p_n)^k = O(1);$$

as $m \longrightarrow \infty$, for $r = 1,2,3,4$.

This completes the proof of lemma.

Proof of Theorem 3.3.4. Since the behaviour of the Fourier series, as for as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of the theorem is a necessary consequence of the lemma 3.3.1.

CHAPTER 4

CHAPTER IV

ON THE GENERALIZED ABSOLUTE NÖRLUND SUMMABILITY

4.1. DEFINITIONS AND NOTATIONS:

Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_o + p_1 + \dots + p_n \neq 0, \qquad n \geq 0.$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \ ;$$

defines the sequence $\{w_n\}$ of the Nörlund means of the sequence $\{s_n\}$, generated by the sequence of coefficient $\{p_n\}$.

The series $\Sigma\ a_n$ is said to be summable $|N,p_n|$, if (Mears [48])

$$\sum_{n=1}^{\infty} |w_n - w_{n-1}| < \infty \ ;$$

and it is said to be summable $|N,p_n|_k$, $k \geq 1$ if (Borwein and Cass [22])

$$\sum_{n=1}^{\infty} n^{k-1} |w_n - w_{n-1}|^k < \infty .$$

In the special case in which

$$p_n = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\,\Gamma(\alpha)} , \ \alpha \geq 0$$

the Nörlund mean reduces to (C,α) mean and $|N,p_n|_k$ reduces to $|C,\alpha|_k$ summability. For $p_n = 1$ and $P_n = n+1$, we get $(C,1)$ mean and $|N,p_n|_k$-summability becomes $|C,1|_k$ summability.

If we take $p_n = \frac{1}{n+1}$, we get absolute Hormonic summabili with index k, symbollically written as $|N, \frac{1}{n+1}|_k$.

4.2. INTRODUCTION:

Concerning $|C,1|$ and $|N,p_n|$ summability Kishore [40] proved the following result.

Theorem 4.2.1 [40]. Let $p_o > 0$, $p_n \geq 0$ and let p_n be a non-increasing sequence. If $\Sigma\, a_n$ is summable $|C,1|$, then the series $\Sigma\, a_n P_n (n+1)^{-1}$ is summable $|N,p_n|$.

In 1972, Ahmad [5] gave the following results related to the absolute Nörlund summability factors of power series and Fourier series.

Theorem 4.2.2 [5]. Let $\{p_n\}$ be as in Theorem 4.2.1.

If

(4.2.1) $$\sum_{v=1}^{n} \frac{1}{v} |t_v| = O(X_n) \quad \text{as} \quad n \longrightarrow \infty$$

where $\{X_n\}$ is a positive non-decresing sequence, and if the sequence $\{\lambda_n\}$ is, such that

(4.2.2) $$X_n \lambda_n = O(1) \ ;$$

(4.2.3) $$n \Delta X_n = O(X_n) \ ;$$

(4.2.4) $$\Sigma \ n \ X_n \ |\Delta^2 \lambda_n| < \infty \ ;$$

then $\Sigma \ a_n \ P_n(n+1)^{-1}$ is summable $|N,p_n|$.

Theorem 4.2.3 [5]. Let $\{p_n\}$ be as in Theorem 3.2.1.

If

$$\lambda_n \log n = O(1) \ ;$$

$$\Sigma \ n \log n \ |\Delta^2 \lambda_n| < \infty \ ;$$

then $\Sigma \ B_n(x) \ P_n \lambda_n (n+1)^{-1}$ is summable $|N,p_n|$ for almost all x.

Theorem 4.2.4 [5]. Let $\{p_n\}$ be as in Theorem 3.2.1. If F is even, $F \in L^2(-\pi,\pi)$,

$$\int_0^t |F(x)|^2 \, dx = O(t) \quad \text{as} \quad t \longrightarrow +0,$$

and if $\{\lambda_n\}$ satisfies the same conditions as in Theorem 4.2.3, then the sequence $\{A_n\}$ of Fourier coefficients of F has the property that $\Sigma A_n P_n \lambda_n (n+1)^{-1}$ is summable $|N,p_n|$.

Theorem 4.2.5 [5]. If $f(z) = \Sigma C_n Z^n$ is a power series of complex class L, such that

$$\int_0^t |f(e^{i\theta})| \, d\theta = O(|t|) \quad \text{as} \quad t \longrightarrow +0,$$

and if $\{\lambda_n\}$ satisfies the conditions as in Theorem 3.2.3, then $\Sigma C_n P_n \lambda_n (n+1)^{-1}$ is summable $|N,p_n|$.

Again, the above mentioned results by Ahmad [5], were established by Bor [18] under weaker conditions and the proof of Bor [18] were shorter and different from Ahmad's. We mention them as follows:

Theorem 4.2.6 [18]. Let $\{p_n\}$ be as in Theorem 3.2.1. Let $\{X_n\}$ be a positive non-decreasing sequence and suppose that there exists sequences $\{\lambda_n\}$ and $\{\beta_n\}$, such that

(4.2.5) $$|\Delta\lambda_n| \leq \beta_n, \quad (\Delta\lambda_n = \lambda_n - \lambda_{n+1})$$

(4.2.6) $$\beta_n \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty ;$$

(4.2.7) $$\sum_{n=1}^{\infty} n X_n |\Delta\beta_n| < \infty,$$

(4.2.8) $$|\lambda_n| X_n = O(1);$$

If

(4.2.9) $$\sum_{v=1}^{n} \frac{1}{v}|t_v|^k = O(X_n); \quad \text{as} \quad n \longrightarrow \infty,$$

then the series $\Sigma\, a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|$.

It may be possible to choose $\{\beta_n\}$ satisfying (4.2.5) so that $\Delta\beta_n$ is much smaller than $|\Delta^2\lambda_n|$. That is, roughly speaking, when $\{\Delta\lambda_n\}$ oscillates it may be possible to choose $\{\beta_n\}$ so that $|\Delta\beta_n|$ is significantly smaller than $|\Delta^2\lambda_n|$ so that $\Sigma\, nX_n|\Delta\beta_n| < \infty$ is a weaker requirement than $\Sigma\, nX_n|\Delta^2\lambda_n| < \infty$. This fact can be verified by the following example:

Take

$$\Delta\lambda_n = \begin{cases} \frac{1}{n(n+1)} & (n \text{ even}) \\ 0 & (n \text{ odd}). \end{cases}$$

Then

$$\Delta^2\lambda_n = \begin{cases} \frac{1}{n(n+1)} & (n \text{ even}), \\ -\frac{1}{(n+1)(n+2)} & (n \text{ odd}). \end{cases}$$

But we take $\beta_n = \frac{1}{n(n+1)}$, so that

$$\Delta\beta_n = \frac{2}{n(n+1)\ (n+2)} \quad .$$

Thus, the condition (4.2.4) of Ahmad [5] is stronger than the condition (4.2.7) of Bor [18].

<u>Theorem 4.2.7 [18]</u>. Let $\{p_n\}$ be as in Theorem 3.2.1. Suppose that $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy conditions (4.2.5) - (4.2.6) of Theorem 4.2.6 and

(4.2.10) $\qquad \lambda_n \log n = O(1)$,

(4.2.11) $\qquad \Sigma\ n \log n\ |\Delta\beta_n| < \infty$.

Then, $\Sigma\ \beta_n(x)\ P_n\lambda_n(n+1)^{-1}$ is summable $|N,p_n|$ for almost all x.

<u>Theorem 4.2.8 [18]</u>. Let $\{p_n\}$ be as in Theorem 4.2.1. If F is even, $F \in L^2(-\pi,\pi)$,

$$\int_0^t |F(x)|^2 \, dx = O(t);$$

as $t \longrightarrow +0$, and if $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the same conditions as in Theorem 4.2.7, then the sequence $\{A_n\}$ of Fourier coefficients of F has the property that $\Sigma A_n P_n \lambda_n (n+1)^{-1}$ is summable $|N,p_n|$.

<u>Theorem 4.2.9 [18]</u>. If $f(z) = \Sigma C_n Z^n$ is a power series of complex class L, such that

$$\int_0^t |f(e^{i\theta})| \, d\theta = O(|t|);$$

as $t \longrightarrow +0$, and if $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the same conditions as in Theorem 4.2.7, then $\Sigma C_n P_n \lambda_n (n+1)^{-1}$ is summable $|N,p_n|$.

Recently, Bor [20] generalized Theorem 4.2.6 for $|N,p_n|_k$, with $k \geq 1$, summability as follows:

<u>Theorem 4.2.10 [20]</u>. Let $\{p_n\}$ be the same as in Theorem 4.2.1 and let $\{X_n\}$ be a positive non-decreasing sequence.

Let the sequences $\{\lambda_n\}$ and $\{\beta_n\}$ are, such that the conditions (4.2.5) - (4.2.8) of Theorem 4.2.6 are satisfied. If

$$(4.2.12) \qquad \sum_{v=1}^{n} \frac{1}{v} |t_v|^k = O(X_n);$$

as $n \longrightarrow \infty$, then the series $\Sigma\, a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|_k$ for $k \geq 1$.

<u>Remark</u>. It may be noted that if we take $k = 1$ in the above theorem, then we get Theorem 4.2.6.

The following results by Verma [74] and Mishra [52] are used respectively for the proof of Theorem 4.2.10.

<u>Lemma 4.2.1 [74].</u> Let $p_o > 0$, $p_n \geq 0$ and let $\{p_n\}$ be a non-increasing sequence. If $\Sigma\, a_n$ is summable $|C,1|_k$, then the series $\Sigma\, a_n P_n (n+1)^{-1}$ is summable $|N, p_n|_k$, $k \geq 1$.

<u>Lemma 4.2.2 [52]</u>. Let $\{X_n\}$ be a positive non-decreasing sequence and the sequences $\{\lambda_n\}$ and $\{\beta_n\}$ are such that conditior (4.2.5) - (4.2.7) of Theorem 4.2.6 are satisfied.

Then,

$$n\beta_n X_n = O(1), \tag{4.2.13}$$

$$\sum_{n=1}^{\infty} X_n \beta_n < \infty . \tag{4.2.14}$$

Proof of Theorem 4.2.10. In order to prove the theorem, we consider only the special case in which (N,p_n) is $(C,1)$, that is, we shall prove that $\Sigma\, a_n \lambda_n$ is summable $|C,1|_k$, $k \geq 1$. The theorem, then follows with the aid of the lemma 4.2.1.

Let $\{T_n\}$ be the n-th (C,1) mean of the sequence $\{n\, a_n \lambda_n\}$. That is to say that

$$T_n = \frac{1}{n+1} \sum_{v=1}^{n} v\, a_v \lambda_v \tag{4.2.15}$$

Now, using Abel's transformation, we have

$$T_n = \frac{1}{n+1} \sum_{v=1}^{n} v\, a_v \lambda_v$$

$$= \frac{1}{n+1} \sum_{v=1}^{n-1} \Delta\lambda_v (v+1) t_v + \lambda_n t_n = T_{n,1} + T_{n,2}, \quad \text{say.}$$

To complete the proof of the theorem, by Minkowski's inequality for $k > 1$, it is sufficient to show that

$$(4.2.16) \qquad \sum_{n=1}^{\infty} \frac{1}{n} |T_{n,r}|^k < \infty ;$$

for $r = 1,2$. Now, applying Hölder's inequality with indices k and k', where

$\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\sum_{n=2}^{m+1} \frac{1}{n} |T_{n,1}|^k \leq \sum_{n=2}^{m+1} \frac{1}{n(n+1)^k} \left\{ \sum_{v=1}^{n-1} |\Delta\lambda_v| (v+1) |t_v| \right\}^k ;$$

$$= \sum_{n=2}^{m+1} \frac{1}{n(n+1)^k} \left\{ \sum_{v=1}^{n-1} |\lambda_v| v \frac{v+1}{v} |t_v| \right\}^k ;$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{k+1}} \left\{ \sum_{v=1}^{n-1} v \beta_v |t_v| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \left\{ \sum_{v=1}^{n-1} (v\beta_v)^k |t_v|^k \right\} x \left\{ \frac{1}{n} \sum_{v=1}^{n-1} 1 \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} (v\beta_v)^k |t_v|^k \sum_{n=v+1}^{m+1} \frac{1}{n^2}$$

$$= O(1) \sum_{v=1}^{m} (v\beta_v)^{k-1} v\beta_v \frac{1}{v} |t_v|^k .$$

Since $v\beta_v = O(1/X_v) = O(1)$ by (4.2.13), we have that

$$\sum_{n=2}^{m+1} \frac{1}{n} |T_{n,1}|^k = O(1) \sum_{v=1}^{m} v\beta_v \frac{1}{v} |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| \sum_{r=1}^{v} \frac{1}{r} |t_r|^k + O(1)\ m\beta_m \sum_{v=1}^{m} \frac{1}{v} |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)|\ X_v + O(1)\ m\ \beta_m\ X_m$$

$$= O(1) \sum_{v=1}^{m-1} v|\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} |\beta_{v+1}| X_v + O(1) m\beta_m X_m$$

$$= O(1)$$

as $m \longrightarrow \infty$, by (4.2.5), (4.2.7), (4.2.9), (4.2.12), (4.2.13) and (4.2.14).

Finally, since $|\lambda_n| = O(1/X_n) = O(1)$ by (4.2.7), we obtain that

$$\sum_{n=1}^{m} \frac{1}{n} |T_{n,2}|^k = \sum_{n=1}^{m} |\lambda_n|^{k-1} |\lambda_n| \frac{1}{n} |t_n|^k$$

$$= O(1) \sum_{n=1}^{m} |\lambda_n| \frac{1}{n} |t_n|^k$$

$$= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{r=1}^{n} \frac{1}{r} |t_r|^k + O(1) |\lambda_m| \sum_{n=1}^{m} \frac{1}{n} |t_n|^k$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n|\ X_n + O(1)\ |\lambda_m|\ X_m$$

$$= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \; |\lambda_m| \; X_m$$

$$= O(1)$$

as $m \longrightarrow \infty$ by virtue of (4.2.5), (4.2.8), (4.2.9), (4.2.12) and (4.2.14).

Therefore, we get that

$$\sum_{n=1}^{m} \frac{1}{n} \left| T_{n,r} \right|^{k} = O(1) \; ;$$

as $m \longrightarrow \infty$, for $r = 1,2$.

This completes the proof of the theorem.

CHAPTER 5

CHAPTER V

$|J,p_n|_k$-SUMMABILITY OF FOURIER SERIES

5.1. DEFINITIONS AND NOTATIONS:

Suppose that $p_n > 0$, $\sum_{n=0}^{\infty} p_n = \infty$;

and that the radius of convergence of the power series

$$p(x) = \Sigma\, p_n x^n \;;\; p(0) = p_0$$

is 1. Given any series $\Sigma\, a_n$, with the sequence of partial sums $\{s_n\}$, we shall use the notations:

$$(5.1.1) \qquad p_s(x) = \sum_{n=0}^{\infty} p_n s_n x^n \;;$$

and

$$(5.1.2) \qquad J(x) = J_s(x) = p_s(x)/p(x).$$

If the series on the right of (5.1.1) is convergent in the right open interval $(C,1)$, $(0 < C < 1)$, and if $J(x) \in BV(C,1)$, we say that the series $\Sigma\, a_n$, or the sequence $\{s_n\}$, is absolutel summable (J,p_n), or simply summable $|J,p_n|$, (see Ahmad [2], [3],

[4]). A series $\Sigma\, a_n$ is summable $|J,p_n|_k$, $k \geq 1$, if $J(x) \in BV^k(C,1)$, that is

$$(2.1.3) \qquad \int_0^1 (1-x)^{k-1} \left|\frac{d}{dx}\left\{J(x)\right\}\right|^k dx < \infty, \quad 0 < C < 1.$$

It is clear that summability $|J,p_n|_1$ is the same as the summability $|J,p_n|$. For $k > 1$, the summability $|J,p_n|$ and $|J,p_n|_k$ are independent of each other (see [45]). Also, for $p_n = 1/n$, $n = 1,2,\ldots$, we get summability $|L|_k$ (see [45]).

We also write, throughout

$$(2.1.4) \qquad g(t) = \int_t^{\pi} \frac{f(u)}{2 \sin u/2}\, du.$$

5.2. INTRODUCTION:

R. Mohanty and J.N. Patnaik [56] have proved the following theorem for an even function f.

Theorem 5.2.1. [56]. If the function

$$(5.2.1) \qquad \frac{1}{t \log(2\pi/t)} \int_t^{\pi} \frac{f(u)}{2 \sin u/2}\, du = \frac{g(t)}{t \log(2\pi/t)}$$

is integrable L in the interval $(0,\pi)$, then the Fourier series of f is $|L|$-summable at the origin.

M. Izumi and S. Izumi [38] generalized this result in a couple of directions for $|J,p_n|$-summability. They proved:

Theorem 5.2.2 [38]. Suppose that

(i) the sequence $\{n\, p_n\}$ is of bounded variation,

(ii) there is an a, $0 < a < 1$, such that

$$(5.2.2) \qquad (1-x)^a\, p(x) \downarrow \quad \text{as} \quad x \uparrow 1.$$

If $\{g(t)/t\; p(1-t)\} \in L(0,\pi)$,

then the Fourier series of f is $|J,p_n|$-summable at the origin.

The condition (5.2.2) may be replaced by that

$$p'(z) = O\left(\frac{1}{|1-z|}\right),\; p''(z) = O\left(\frac{1}{|1-z|^2}\right), \quad \text{as} \quad z \longrightarrow 1,$$

where $z = x\, e^{it}$ and $p(z) = \Sigma\, p_n z^n$.

If $p_n = \frac{1}{n}$ $(n = 1,2,3,\ldots)$, then Theorem 5.2.2 reduces to Theorem 5.2.1.

<u>Theorem 5.2.3 [38]</u>. Suppose that

(i) $\{n\, p_n\}$ and $\{n^2\, p_n\}$ are monotone and concave and convex, and that

(ii) $(1-x)^2\, p''(x)/p(x) \in L(0,\pi)$.

If

$$(5.2.3) \qquad \int_0^1 G(t)\, t^{-3}\, dt \int_{1-t}^1 \left\{(1-x)^2\, p''(x)/p(x)\right\} dx < \infty ;$$

where

$$G(t) = \int_0^t |g(u)|\, du ;$$

then the Fourier series of f is $|J,p_n|$-summable at the point x_0.

Ahmad and Rahiman [7] proved a couple of theorems as follows:

<u>Theorem 5.2.4 [7]</u>. Suppose that

(i) the sequence $\{p_n\}$ is of bounded variation, and that

$$(5.2.4) \qquad (1-x)^a\, p(x) \downarrow \text{ as } x \uparrow 1.$$

If

$$\left\{ g(t)/t^{2} p(1-t) \right\} \in L(0,\pi) \ ;$$

then the Fourier series of f is $|J,p_n|$-summable at the origin.

The condition (i) may be replaced by that

(5.2.5) $\quad p(z) = O\left(\frac{1}{|1-z|}\right), \ p'(z) = O\left(\frac{1}{|1-z|^{2}}\right), \ p''(z) = \left(\frac{1}{|1-z|^{3}}\right.$

as $z \longrightarrow 1$, where $z = x e^{it}$ and $p(z) = \Sigma\, p_n z^{n}$.

Theorem 5.2.5 [7]. Suppose that

(i) the sequence $\{p_n\}$ is of bounded variation

(ii) there is an a, $0 < a < 1$, such that

$$(1-z)^{a} p(x) \downarrow \text{ as } x \uparrow 1 \ ;$$

and, for $z = x e^{it}$, as $z \longrightarrow 1$;

(iii) $p'(z) = O(p(z))$;

(iv) $(1-z)\, p''(z) = O(p(z))$.

If

$\left\{ g(t)/t \; p(1-t) \right\} \in L(0,\pi)$, then the Fourier series of f is $|J,p_n|$- summable at the origin.

The condition (i) may be replaced by that

$$(5.2.6) \qquad p(z) = O\left(\frac{1}{|1-z|}\right), \quad \text{as} \quad z \longrightarrow 1.$$

It is observed that Theorem 5.2.4 gives a partial generalization of Theorem 5.2.2 and yields a criterian for absolute Abel summability for Fourier series, while Theorem 5.2.5 gives a complete generalization of Theorem 5.2.2 and contains Theorem 5.2.1 as a special case when $p_n = \frac{1}{n}$, $n = 1,2,\ldots$.

Again, Ahmad and Rahiman [8] proved couple of correspondi theorems for summability $|J,p_n|_k$, $k \geq 1$, of Fourier seires whic contains Theorems 5.2.4 and 5.2.5 as special cases (see also [68] as follows:

<u>Theorem 5.2.6 [8]</u>. Let $k \geq 1$. Suppose that

(i) the sequence $\{p_n\}$ is of bounded variation

(ii) there is an a, $k < a < 2k$, such that

$$(1-x)^{a/k}\, p(x) \downarrow \quad \text{as} \quad x \uparrow 1 \;;$$

and that,

(iii) for $k > 1$, the function $\{(1-x)\; p(x)\}^{1-k}$ is bounded in $(C,1)$, $0 < C < 1$.

If

$$\left\{ g(t)/t^2 \; p(1-t) \right\} \in L(0,\pi) \;;$$

then the Fourier series of f is $|J,p_n|_k$-summable at the origin.

The condition (i) may be replaced by that

$$(5.2.7) \quad p(z) = O\left(\frac{1}{|1-z|}\right), \; p'(z) = O\left(\frac{1}{|1-z|^2}\right), \; p''(z) = O\left(\frac{1}{|1-z|^3}\right);$$

as $z \longrightarrow 1$, where $z = x\,e^{it}$ and $p(z) = \Sigma\, p_n z^n$.

Theorem 5.2.7 [8]. Let $k \geq 1$, suppose that

(i) the sequence $\{p_n\}$ is of bounded variation

(ii) there is an a, $0 < a/k < 1$, such that

$$(1-x)^{a/k} \; p(x) \downarrow \quad \text{as} \quad x \uparrow 1 \;;$$

and for $z = x\,e^{it}$, as $z \longrightarrow 1$.

(iii) $p'(z) = O\,(p(z))$, and

(iv) $(1-z)\; p''(z) = O(p(z))$.

If

$$\left\{ g(t)/t \; p(1-t) \right\} \in L(0,\pi);$$

then the Fourier series of f is $|J,p_n|_k$-summable at the origi

The condition (i) may be replaced by that

$$(5.2.8) \qquad p(z) = O\left(\frac{1}{|1-z|}\right).$$

It is important to note that, in the special cases when $p_n = A_n^{\alpha}$, $-1 < \alpha \leq 0$, for $n = 0,1,2,\ldots$, and $p_n = \frac{1}{n}$, for $n = 1,2,\ldots$, we get respectively the following interesting results from the above Theorems 5.2.6 and 5.2.7.

<u>Corallary 5.2.1</u>. Let $k \geq 1$ and $-1 < \alpha \leq 0$. If

$$\left\{ g(t)/t^{1-\alpha} \right\} \in L(0,\pi);$$

then the Fourier series of f is $|A_\alpha|_k$-summable at the origin.

<u>Corallary 5.2.2</u>. Let $k \geq 1$. If

$$\left\{ \frac{g(t)}{t \log\left(\frac{2\pi}{t}\right)} \right\} \in L(0,\pi);$$

then the Fourier series of f is $|L|_k$-summable at the origin.

Subsequently, Rahiman ([67]; see also [68]) generalized Theorem 5.2.3 for $|J,p_n|_k$ to give the following theorem:

Theorem 5.2.8 [67]. Let $k \geq 1$. Suppose that

(i) $\{n\, p_n\}$ and $\{n^2\, p_n\}$ are monotone and concave or convex;

(ii) $\{(1-x)^{3-1/k}\, p''(x)/p(x)\} \in L^k(0,1)$.

If

$$(5.2.9) \qquad \int_0^1 \frac{G^k(t)}{2^{2k+1}}\, dt \int_{1-t}^1 (1-x)^{3k-1} \left(\frac{p''(x)}{p(x)}\right)^k < \infty ;$$

where

$$G(t) = \left\{ \int_0^t |g(u)|^k\, du \right\}^{1/k} ;$$

then the Fourier series of f is $|J,p_n|_k$-summable at the point x_0.

The condition (5.2.9) is satisfied when

$$(5.2.10) \qquad \left(\int_0^t u^{3k-1} \left(\frac{p''(1-u)}{p(1-u)}\right)^k du\right)^{1/k} \leq \frac{t^3\, p''(1-t)}{p(1-t)} ;$$

for all $t > 0$; and

$$(5.2.11) \qquad t^{1-1/k}\, \frac{p''(1-t)}{p(1-t)}\, G(t) \in L^k(0,1).$$

For the proof of the theorem 5.2.8 following lemma is needed.

<u>Lemma 5.2.1</u>. For $0 < C < 1$,

$$\left| \sum_{r=1}^{\infty} n\, p_n \cos(n+1/2)\ t(x^n/p(x))' \right| \leq \frac{K(1-x)^2\, p''(x)}{(1-x)^2 + t^2 p(x)}$$

on the interval $(C,1)$, where K is a constant.

<u>Proof of the theorem 5.2.8</u>. Since the theorem is true for $k = 1$ (see Theorem 5.2.3), we proceed to prove it for $k > 1$. We can suppose that

$$(5.2.12) \qquad \int_0^{\pi} \phi(n)\, du = 0 \ ;$$

and

$$p_1 = p_2 = 0.$$

The sequence $\{n\, p_n \ ;\ n \geq 3\}$ is also monotone and concave or convex.

Let $S_n(x_o)$ be the n-th partial sum of the Fourier series of f at the point x_o, then

$$S_n(x_o) = \frac{1}{\pi} \int_0^{\pi} \phi(t) \frac{\sin\,(n+1/2)t}{2 \sin t/2}\, dt.$$

Therefore,

$$J(x) = \frac{1}{p(x)} \sum_{n=1}^{\infty} p_n s_n(x_o) x^n$$

$$= \frac{1}{\pi p(x)} \int_o^{\pi} \frac{\phi(t)}{2 \sin t/2} \left(\sum_{n=1}^{\infty} p_n \operatorname{Sin} (n+1/2) t \, x^n \right) dt.$$

Differentiating with respect to x, we get

$$J'(x) = \frac{1}{\pi} \int_o^{\pi} \frac{\phi(t)}{2 \operatorname{Sin} t/2} \left(\sum_{n=1}^{\infty} p_n \operatorname{Sin} (n+\tfrac{1}{2}) t (x^n/p(x))' \right) dt$$

$$(5.2.13) \quad = \frac{1}{\pi} \int_o^{\pi} g(t) \left(\sum_{n=1}^{\infty} (n+1/2) p_n \cos(n+1/2) t (x^n/p(x))' \right) dt \, ;$$

where $'$ denotes the differentiation with respect to x.

Write

$$J'(x) = \frac{1}{\pi} \int_o^{\pi} g(t) \, F'(x,t) \, dt;$$

where

$$F'(x,t) = \sum_{n=1}^{\infty} (n+1/2) p_n \cos(n+1/2) t (x^n/p(x))' .$$

Now, the Fourier series of f is summable $|J,p_n|_k$, if

$$(5.2.14) \qquad \int_c^1 (1-x)^{k-1} \, |J'(x)|^k < \infty .$$

Since

$$J'(x) = \frac{1}{\pi} \int_0^{\pi} g(t)\, F'(x,t)\, dt$$

$$= \frac{1}{\pi} \int_0^{\pi} g(t)\, F_1'(x,t)\, dt + \frac{1}{\pi} \int_0^{\pi} g(t)\, F_2'(x,t)\, dt$$

$$= J_1'(x) + J_2'(x), \text{ say.}$$

Where

$$F_1'(x,t) = \sum_{n=1}^{\infty} n\, p_n \cos(n+1/2)\, t(x^n/p(x))'$$

and

$$F_2'(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} p_n \cos(n+1/2)\, t(x^n/p(x))',$$

in order to prove (5.2.14), by Minkowski's inequality, it is enough to show that

$$(5.2.15) \qquad I_r = \int_c^1 (1-x)^{k-1}\, |J_r'(x)|^k\, dx < \infty, \; r = 1,2.$$

Proof of (5.2.15). We have

$$I_1 = \int_c^1 (1-x)^{k-1}\, |J_1'(x)|^k\, dx$$

$$\leq \int_c^1 (1-x)^{k-1} \left(\frac{1}{\pi} \int_0^{\pi} |g(t)|\,|F_1'(x,t)|\,dt\right)^k dx$$

$$\leq \left(\frac{1}{\pi}\right)^k \int_c^1 dx \int_0^{\pi} (1-x)^{k-1} |g(t)|^k\, |F_1'(x,t)|^k\, dt \left(\int_0^{\pi} dt\right)^{k-1}$$

$$= \frac{\pi^{k-1}}{\pi^k} \int_c^1 dx \int_0^\pi (1-x)^{k-1} \; |g(t)|^k \; |F_1'(x,t)|^k \; dt$$

$$\leq \frac{1}{\pi} \int_0^\pi |g(t)|^k \; dt \int_c^1 (1-x)^{k-1} \; |F_1'(x,t)|^k \; dx$$

$$= \frac{1}{\pi} \int_0^{1-c} |g(t)|^k \; dt \int_c^{1-t} (1-x)^{k-1} \; |F_1'(x,t)|^k \; dx$$

$$+ \frac{1}{\pi} \int_0^{1-c} |g(t)|^k \; dt \int_{1-t}^1 (1-x)^{k-1} \; |F_1'(x,t)|^k \; dx$$

$$+ \frac{1}{\pi} \int_{1-c}^\pi |g(t)|^k \; dt \int_c^1 (1-x)^{k-1} \; |F_1'(x,t)|^k \; dx$$

$$= I_{11} + I_{12} + I_{13}, \quad \text{say.}$$

Now, we see that

$$I_{11} = \frac{1}{\pi} \int_0^{1-c} |g(t)|^k \; dt \int_c^{1-t} (1-x)^{k-1} \; |F_1'(x,t)|^k \; dx$$

$$\leq \frac{K^*}{\pi} \int_0^{1-c} |g(t)|^k \; dt \int_c^{1-t} (1-x)^{k-1} \left\{\left(\frac{p''(x)}{p(x)}\right)\right\}^k dx$$

where K^* denotes a constant not necessarily the same at each occurance.

$$= K \int_c^1 (1-x)^{k-1} \left(\frac{p''(x)}{p(x)}\right)^k dx \left(\int_c^{1-x} |g(t)|^k \; dt\right)^{1/k}$$

$$= K \int_0^1 (1-x)^{k-1} \left(\frac{p''(x)}{p(x)}\right)^k (G(1-x))^k dx$$

$$= - K \int_{1-c}^{o} t^{k-1} \left(\frac{p''(1-t)}{p(1-t)}\right)^k G^k(t)\, dt$$

$$= K \int_o^{1-c} t^{k-1} \left(\frac{p''(1-t)}{p(1-t)}\right)^k G^k(t) dt$$

$$\leq K \int_o^1 t^{k-1} \left(\frac{p''(1-t)}{p(1-t)}\right)^k G^k(t) dt$$

(5.2.16) $\leq$ K, using (5.2.11).

Next,

$$I_{12} = \frac{1}{\pi} \int_o^{1-c} |g(t)|^k dt \int_{1-t}^{1} (1-x)^{k-1} |F_1'(x,t)|^k dx$$

$$\leq \frac{K}{\pi} \int_o^{1-c} \frac{|g(t)|^k}{t^{2k}} dx \int_{1-t}^{1} (1-x)^{3k-1} \left(\frac{p''(x)}{p(x)}\right)^k dx \text{ (see lemma 5.2.1)}$$

$$= K \int_c^1 (1-x)^{3k-1} \left(\frac{p''(x)}{p(x)}\right)^k dx \int_{1-x}^{1-c} \frac{|g^k(t)|^k}{t^{2k}} dt$$

$$= K \int_c^1 (1-x)^{3k-1} \left(\frac{p''(x)}{p(x)}\right)^k dx \int_{1-x}^{1-c} \frac{G^k(t)}{t^{2k-1}} dt$$

$$= K \int_o^{1-c} \frac{G^k(t)}{t^{2k+1}} dt \int_{1-t}^{1} (1-x)^{3k-1} \left(\frac{p^k(x)}{p(x)}\right)^k dx$$

(5.2.17) $\leq$ K ; by using the hypothesis (5.2.9).

Finally,

$$I_{13} = \frac{1}{\pi} \int_{1-c}^{\pi} |g(t)|^{k} dt \int_{c}^{1} (1-x)^{k-1} |F_1'(x,t)|^{k} dx$$

$$\leq \frac{K}{\pi} \int_{1-c}^{\pi} |g(t)|^{k} dt \int_{c}^{1} (1-x)^{3k-1} \left(\frac{p''(x)}{p(x)}\right)^{k} dx$$

(5.2.18) $\leq K \left\{ G(\pi) \right\}^{k} \leq K,$

again by hypothesis of the theorem. Combining the inequalities (5.2.16), (5.2.17) and (5.2.18), we get

$$I_1 \leq K.$$

Similarly, I_2 is also finite and the proof of the theorem is complete.

BIBLIOGRAPHY

Abel. N.H.

[1] : Untersuchungen Über die Reihe $1+mx+\frac{m(m-1)}{1.2}x^2+\ldots$, J. für die reine und angewandte Mathematik, (Crelles), 1 (1826), 311-339.

Ahmad, Z.U.

[2] : Contributions to the study of absolute summability, D.Sc. Thesis, University of Jabalpur, 1967.

[3] : On inclusion among some absolute summability methods Alig. Bull. Math., I (1971), 31-37.

[4] : On absolute summability methods based on power series I, Rend. Math. (6) 5 (1972), 541-549.

[5] : Absolute Nörlund summability factors of power series and Fourier series; Ann. Polon. Math. 27 (1972), 9-20.

[6] : Absolute summability of Fourier series, its conjugate series and their derived series by methods based on power series, Allahabad Mathematical Society, Second Biennial Conference (April 7-9, 1990)

Ahmad, Z.U., and Rahiman, Abdul P.M.

[7] : Absolute summability of Fourier series by methods based on power series, Indian Jour. Math., 17 (1975), 71-80.

Ahmad, Z.U., and Rahiman, Abdul P.M.

[8] : On $|J,p_n|_k$-summability of Fourier series I, Alig. Bull. Math., 5-6 (1975-76), 25-36.

[9] : Tauberian Theorems for $|J,p_n|$-summability, Abstract, Notices of the American Math. Soc. (1976).

Ahmad, Z.U., and Varshney, K.C.

[10] : Tauberian Theorems for $|J,p_n|$-summability, Soochow J. Math., 9 (1983), 1-16.

Bhatt, S.N.

[11] : An aspect of local property of $|R \log n,1|$-summabilit of Factored Fourier Series, Tôhoku Math. J. (2); 11 (1959), 13-19.

[12] : An aspect of local property of $|R, \log n,1|$-summability of Factored Fourier Series, Proc. Nat. Inst. India 26 (1960), 69-73.

[13] : An aspect of local property of $|N,p_n|$-summability of Fourier Series, Indian J. Math., 5 (1963), 87-91.

Bor, H.

[14] : A note on two summability methods, Proc. Amer. Math. Soc., 98 (1980), 81-84.

[15] : On $|\bar{N},p_n|_k$-summability factors of infinite series, Tamkang J. Math., 16(1) 1985, 13-20.

[16] : On $|\bar{N},p_n|$-summability factors of infinite series, Proc. Indian Acad. Sci. (Math. Sci.), 98 (1988), 53-57.

Bor, H.

[17] : Local property of $|\overline{N},p_n|_k$-summability of factored Fourier series, Bull. Inst. Math. Acad. Sinica 17 (1989), 165-170.

[18] : Absolute Nörlund summability factors of power series and Fourier series, Ann. Polon. Math. 56 (1991), 11-17.

[19] : On the $|\overline{N},p_n|_k$-summability factors for infinite series, Proc. Indian Acad. Sci. (Math. Sci.), 101 (2) 1991, 143-146.

[20] : On the absolute Nörlund summability factors, Glasnik Mathematicki, 27 (47), 1992, 57-62.

[21] : On the local property of $|\overline{N},p_n|$-summability of Factored Fourier series, J. Math. Analysis and applications, 163 (1) 1992, 220-226.

Borwein, D. and Cass, F.P.

[22] : Strong Nörlund summability, Math. Zeith. 103 (1968), 94-111.

Bosanquent, L.S.

[23] : The absolute Cesàro summability of a Fourier series, Proc. London Math. Soc., (2), 41, (1936), 517-528.

Bosanquent, L.S. and Kestelman, H.

[24] : The absolute convergence of series of integrals, Proc. London Math. Soc., (2), 45 (1939), 88-97.

Cauchy, A.L.
[25] : Caurs d' Analyse Algebriques, Paris, 1821.

Cheng, M.T.
[26] : Summability factors of Fourier Series, Duke Math. J., 15 (1948), 17-27.

Chow, H.C.
[27] : Note on convergence and summability factors, J. London Math. Soc., 20 (1954), 459-476.

[28] : On the summability factors of Fourier series, J. London Math. Soc., 16 (1941), 215-220.

Cooke, R.G.
[29] : Infinite matrices and sequence spaces, Macmillan, 1950, Dover, 1955.

Das, G.
[30] : On some methods of summability, Quart. Jour. Math. Oxford (2), 17 (1966), 244-256.

Fekete, M.
[31] : Zur theorie die divergenten Reihen, Math. és Term Ert. Budapest, 29 (1911), 719-726.

Flett, T.M.
[32] : On an extension of absolute summability and some theorems of Littlewood and Poley, Proc. London Math. Soc. 7 (1957), 113-141.

[33] : Some more theorems concerning the absolute summabilit of Fourier series and Power series, Ibid,(3) 8 (1958) 357-387.

Hardy, G.H.
[34] : Divergent series, Oxford, 1949.

Hardy, G.H; Littlewood, J.E. and Polya, G.
[35] : Inequalities, Cambridge, 1952.

Hsiang, F.C.
[36] : On $|C,1|$-summability factors of Fourier series at a given point, Pacific J. Math., 33 (1970), 139-147.

Hyslop, J.M.
[37] : A Tauberian theorem for absolute summability, Jour. London Math. Soc., 12 (1937), 176-180.

Izumi, M. and Izumi, S.
[38] : Absolute summability of logarithmic method of Fourier series, Proc. Japon Acad., 46 (1970), 656-659.

Jurkat, W. and Peyerimhoff, A.
[39] : Lokalisation bei absoluter Cesàro summerbarkeit von Potenzseihn und trignometrischen Reihen II, Math. Zeitschr, 64 (1956), 151-158.

Kishore, N.
[40] : On the absolute Nörlund summability factors, Riv. Math. Univ. Parma (2) 6 (1965), 129-134.

Knoff, K. and Lorentz, G.G.
[41] : Beitrage Zur absolution Limiterunge, Arch. Math., 2 (1949), 10-16.

Kogbetliantz, E.

[42] : Sur les séries absolument sommables par le méthode des moyennes arithmétiques, Bull. Sci. Math. (1925), 234-256.

Matsumoto, K.

[43] : Local property of the summability $|R, \lambda_n, 1|$, Tôhoku Math. J., (2) 8 (1956), 114-124.

Mazhar, S.M.

[44] : On an extension of absolute Riesz summability, Proc. Nat. Inst. Sci. India, 26 (1960), 160-167.

[45] : On $|L|_k$-summability of Fourier series, Comment. Math. Univ. St. Paul, 20 (1971), 1-8.

[46] : On the absolute summability factors of infinite series, Tôhaku Math. Journ. 23 (1971), 433-451.

[47] : On $|C,\beta|_k$-summability factors of infinite series, Acad. Roy. Belg. Bull. Cl. Sci., 57 (1971), 275-286.

Mears, F.M.

[48] : Some multiplication theorems for the Nörlund means, Bull. Amer. Math. Soc., 41 (1935), 875-880.

[49] : Absolute regularity and the Nörlund means, Annals of Math. 38 (1937), 594-601.

Mehdi, M.R.

[50] : Linear transformations between the Banach spaces L^p and l^p with applications to absolute summability, Ph.D. Thesis, 1959.

Mehdi, M.R.

[51] : Summability factors for generalized absolute summability I, Proc. London Math. Soc. (3) 10 (1960), 180-200.

Mishra, K.N.

[52] : On the absolute Nörlund summability factors of infinite series, Indian J. Pure Appl. Math.,14 (1983), 40-43.

[53] : Multipliers for $|\bar{N},p_n|$-summability of Fourier series, Bull. Inst. Math. Acad. Sinica 14 (1986), 431-438.

Mohanty, R.

[54] : On the summability $|R, \log w, 1|$ of Fourier series, J. London Math. Soc. 25 (1950), 67-72.

[55] : On the summability $|C,1|$ of Fourier series, Bull. Calcutta Math. Soc., 47 (1955), 53-54.

Mohanty, R. and Patnaik, J.N.

[56] : On the absolute L-summability of a Fourier series, Jour. London Math. Soc. 43 (1968), 452-456.

Mahapatra, R.N.

[57] : On absolute Riesz summability factors, J. Indian Math. Soc. 32 (1968), 113-129.

Nörlund, N.E.

[58] : Sur une application des functions permutables, Lunds Universitets Arsakrift (2), 16(1919), No.3.

Pandey, G.S.

[59] : Multipliers for |C,1|-summability of Fourier series, Pacific J. Math. 79(1), (1978), 177-182.

Pati, T.

[60] : On the absolute summability of the conjugate series of a Fourier series, Proc. Amer. Math. Soc. 3 (1952), 852-857.

[61] : Certain methods of absolute summability and their application to Fourier series, Presidential address, Section of Mathematics, 59th Session of ISCA, 1972.

Peyerimhoff, A.

[62] : On convergence fields of Nörlund means, Proc. Amer. Math. Soc. 7 (1956), 335-347.

Prasad, B.N.

[63] : The absolute summability (A) of Fourier series, Proc. Edinburgh Math. Soc. (2) 2 (1930), 129-134.

[64] : On the summability factors of Fourier series and the bounded variation of power series, Proc. London Math. Soc. (2) 35 (1933), 407-424.

Randels, W.C.

[65] : On the absolute summability of Fourier series, Bull. Amer. Math. Soc. 4 (1938), 733-736.

[66] : On the absolute summability of Fourier series II, Bull. Amer. Math. Soc. 46 (1940), 86-88.

Rahiman, Abdul, P.M.

[67] : On $|J,p_n|_k$-summability of Fourier series II, Alig. Bull. Math., 2 (1972), 89-96.

[68] : Certain problems on the absolute summability methods based on power series, Ph.D. Thesis, Aligarh, 1974.

Rizvi, S.M.

[69] : Studies on some aspects of summability and absolute summability, Ph.D. Thesis, Aligarh, 1976.

Schur, I.

[70] : 'Über lineare Transformationen in der Theorie der unendlichen Reihen', J. reine angew Math. 151 (1921), 79-111.

Sunouchi, G.

[71] : Notes on Fourier analysis (XVIII), Absolute summability of series with constant terms, Tôhoku Math. Jour. (2), 1 (1949), 57-65.

Sulaiman, W.T.

[72] : Multipliers for $|C,\delta|_k$-summability of Fourier series, Tamkang J. Math., 21 (3), 1990, 185-190.

Titchmarsh, E.T.

[73] : The Theory of functions, Oxford, 1939.

Verma, S.M.

[74] : On the absolute Nörlund summability factors, Riv. Math. Univ. Parma (4) 3 (1977), 27-33.

Whittakar, J.M.

[75] : The absolute summability Fourier series, Proc. Edinburg Math. Soc. (2) 2 (1930), 1-5.

Woronoi, G.F.

[76] : Extension of the notion of the limit of the sum of terms of an infinite series, Annals of Math., (2) 33 (1932), 422-428.

Zygmond, A.

[77] : Trignometric series, Cambridge University Press, 1959.